C.11 · Integration by parts, order reversal and symmetry

Original classroom teaching; not an official form or score predictor

Integration by parts, order reversal and symmetry

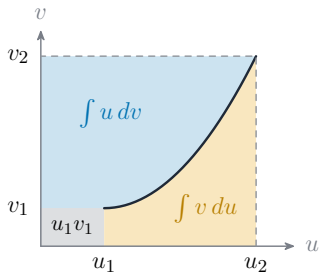
- Use **integration by parts 分部积分**, including repeated and cyclic cases.
- Reverse the order of a double integral by redrawing the region.
- Use **reflection symmetry 反射对称** and odd–even symmetry to simplify definite integrals.

Integration by parts

- It comes from the product rule. Choose u to become simpler when differentiated (logs, powers) and dv easy to integrate.
- Repeated parts: $\int x^2 e^x \, dx = (x^2 - 2x + 2)e^x + C$.
- Cyclic case: for $\int e^x \sin x \, dx$, two rounds of parts give back the original integral; solve for it: $\frac{1}{2}e^x(\sin x - \cos x) + C$.

$$\int u \, dv = uv - \int v \, du$$

Figure: Integration by parts



$$u_2v_2 - u_1v_1 = \int u\,dv + \int v\,du$$

- Integration by parts as two areas.
- Read the axes and units first.

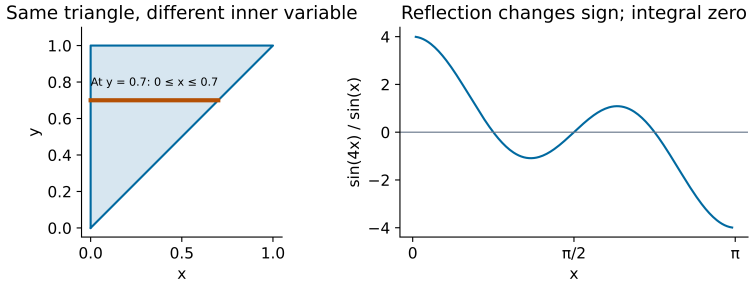
Reversing the order

- Sketch the region described by the limits.
- Read the new limits from the sketch: the outer variable gets constant limits, the inner one runs between curves.
- Fubini: for a continuous function on a bounded closed region (or when $\iint |f|$ is finite), both orders give the same answer.

Symmetry

- If f is odd, $\int_{-a}^{a} f = 0$. If f is even, $\int_{-a}^{a} f = 2\int_{0}^{a} f$.
- Reflection about the midpoint: $\int_{a}^{b} f(x)\,dx = \int_{a}^{b} f(a+b-x)\,dx$.
- If $f(a+b-x) = -f(x)$, the integral is 0. Adding the two forms can also produce an easy integral.

Figure: Symmetry



- Swap the order of integration by reading the limits from the same region; symmetry can make an integral zero
- Match each part of the diagram to the text.

Worked example: reversing and parts

- $\displaystyle\int_0^1\int_x^1 e^{y^2}\,dy\,dx$: the region is $0 \le x \le y \le 1$. Reversed, $\displaystyle\int_0^1\int_0^y e^{y^2}\,dx\,dy = \int_0^1 y e^{y^2}\,dy = \frac{e-1}{2}$.
- $\displaystyle\int_1^x \ln t\,dt = x\ln x - x + 1$, by parts with $u = \ln t$.
- $\displaystyle\int_1^x (\ln t)^2\,dt = x(\ln x)^2 - 2\int_1^x \ln t\,dt = x\left[(\ln x)^2 - 2\ln x + 2\right] - 2$. Check: it is 0 at $x = 1$, and its derivative is $(\ln x)^2$.

Worked example: symmetry

- $I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx$. Replacing x by $\pi/2 - x$ gives $\int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\,dx$, the same I. Adding: $2I = \pi/2$, so $I = \pi/4$.
- $\int_0^{\pi} \frac{\sin 4x}{\sin x}\,dx$: replacing x by $\pi - x$ changes the sign of the integrand, so the integral is 0.
- $\int_{-1}^{1} x^3 \cos x\,dx = 0$, since the integrand is odd.

Common errors

- Choosing u so that the new integral is harder.
- Dropping the boundary term uv in a definite integral.
- Swapping the limits' numbers without redrawing the region.

Common errors (2)

- Swapping the order where Fubini's conditions fail.
- Calling a function odd without checking $f(-x) = -f(x)$.

Key terms

- **integration by parts**: The identity $\int u\,dv = uv - \int v\,du$, derived from the product rule.
- **reflection symmetry**: A relation between the values of an integrand at points mirrored across the midpoint of the interval.

Check yourself

- What is $\int xe^x\,dx$?
- Reverse the order: $\displaystyle\int_0^1 \int_y^1 f(x,y)\,dx\,dy$.
- What is $\displaystyle\int_{-2}^{2} x^5 e^{x^2}\,dx$?