A.7 · LU factorisation and nullity of composed maps

Original classroom teaching; not an official form or score predictor

LU factorisation and nullity of composed maps

- Find and use an LU factorisation, with **forward substitution** 前代 and back substitution.
- Place the permutation correctly when $PA = LU$.
- Read the determinant from the factors.
- Bound the rank and **nullity** 零度 of a composition.

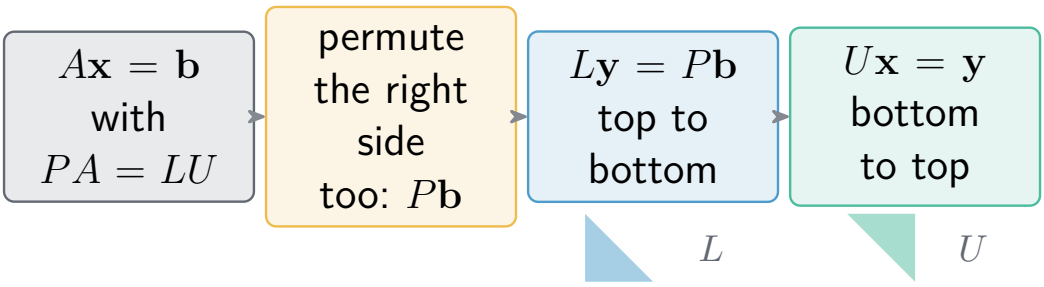
LU factorisation

- U is the upper triangular matrix that elimination produces.
- L is lower triangular with 1s on the diagonal; below the diagonal it holds the multipliers used.
- If a zero pivot appears, swap rows: $PA = LU$, where P is a permutation matrix.

Solving with the factors

- Permute the right side: compute Pb.
- **Forward substitution**: solve $Ly = Pb$ from the top down.
- Back substitution: solve $Ux = y$ from the bottom up.
- $\det A = \pm \prod u_{ii}$, the sign set by the number of row swaps.

Figure: Solving with the factors



- Permute, forward, back.
- Match each part of the diagram to the text.

Rank and nullity of compositions

- $\ker A \subseteq \ker(BA)$, so $\operatorname{nullity}(BA) \geq \operatorname{nullity}(A)$.
- $\operatorname{nullity}(BA) = \operatorname{nullity}(A) + \dim(\operatorname{im} A \cap \ker B) \leq \operatorname{nullity}(A) + \operatorname{nullity}(B)$.
- $\operatorname{rank}(BA) \leq \min(\operatorname{rank} A, \operatorname{rank} B)$.

Worked example: a 2×2 solve

- $Ly = b$: $y_1 = 5$; $2y_1 + y_2 = 12$, so $y_2 = 2$.
- $Ux = y$: $2x_2 = 2$, so $x_2 = 1$; $3x_1 + 1 = 5$, so $x_1 = \frac{4}{3}$.
- Check: $A = LU = \begin{pmatrix} 3 & 1 \\ 6 & 4 \end{pmatrix}$ sends $(\frac{4}{3}, 1)$ to $(5, 12)$, and $\det A = 3 \times 2 = 6$.

Figure: Worked example: a 2×2 solve

LU solve: forward through L, then backward through U

Stage	Known right side	Result
Ly = b	b = (−1, −1, 12)	y = (−1, 1, 8)
Ux = y	y = (−1, 1, 8)	x = (1, −1, 2)

- Solve with LU by **forward substitution** through L, then back substitution through U
- Match each part of the diagram to the text.

Worked example: nullity bounds

- $\operatorname{nullity} A = 5 - 3 = 2$ and $\operatorname{nullity} B = 4 - 3 = 1$.
- $\operatorname{nullity}(BA) = 2 + \dim(\operatorname{im} A \cap \ker B)$, which is 2 or 3.
- So $\operatorname{rank}(BA) = 5 - \operatorname{nullity}(BA)$ is 3 or 2. Without more information, both are possible.

Common errors

- Solving $Ly = b$ instead of $Ly = Pb$ when rows were swapped.
- Substituting in the wrong direction through L or U.
- Forgetting the sign from row swaps in the determinant.

Common errors (2)

- Adding nullities as if equality always held.
- Thinking $\operatorname{rank}(BA)$ can exceed the rank of either factor.

Key terms

- **forward substitution**: Solving a lower triangular system from the first equation downward.
- **nullity**: The dimension of the kernel of a linear map.

Check yourself

- $A = LU$ with U having diagonal 2, 3, 5 and no row swaps. What is $\det A$?
- $A : \mathbb{R}^4 \to \mathbb{R}^4$ has nullity 1, and so does B. What is the largest possible nullity of BA?
- In $PA = LU$, what do you apply P to when solving $Ax = b$?