C.10 · Implicit differentiation and the inverse Jacobian

Original classroom teaching; not an official form or score predictor

Implicit differentiation and the inverse Jacobian

- Find dy/dx for a curve given implicitly.
- Know when the implicit function theorem applies.
- Compute a **Jacobian matrix 雅可比矩阵** and use the inverse function theorem.
- Find the partial derivatives of a **local inverse 局部逆映射**.

Implicit differentiation

- This needs $F_y \neq 0$ at the point. The implicit function theorem guarantees a local function $y(x)$ there if F has continuous partial derivatives.
- Where $F_y = 0$, the tangent may be vertical, or the curve may cross itself.

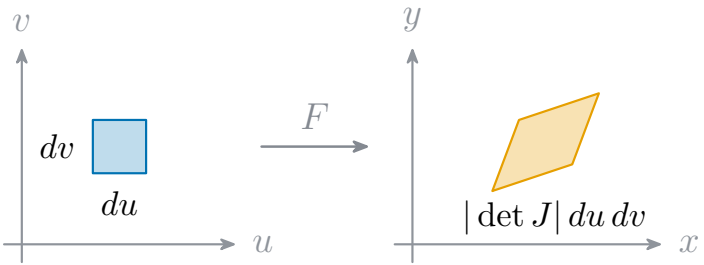
$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

The Jacobian matrix

- $|\det J|$ is the factor by which small areas are scaled.
- A small $du\,dv$ square is mapped to a parallelogram of area $|\det J|\,du\,dv$.

$$J = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

Figure: The Jacobian matrix



$$J = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

to first order

- The Jacobian scales small areas.
- Match each part of the diagram to the text.

The inverse function theorem

- Invert the whole matrix. In general $u_x \neq 1/x_u$, because when x changes with y fixed, both u and v must move.
- $\det(J^{-1}) = 1/\det J$.

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = J^{-1}$$

Figure: The inverse function theorem

Coupled sensitivity: inverse entries differ from reciprocals

Forward map	Jacobian	Inverse Jacobian
$x = u + v;\ y = u + 2v$	[[1, 1], [1, 2]]	[[2, −1], [−1, 1]]
$x = u^2 + v;\ y = u - v$ at $(u, v) = (1, 0)$	[[2, 1], [1, −1]]	[[1/3, 1/3], [1/3, −2/3]]

- Invert the whole Jacobian matrix; its entries are not the reciprocals of the forward entries
- Match each part of the diagram to the text.

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Worked example: inverse partials

- $J = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$, with $\det J = -3 \neq 0$.
- $J^{-1} = \frac{1}{-3}\begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{pmatrix}$.
- So $u_x = \frac{1}{3}$, not $1/x_u = \frac{1}{2}$. Check: $JJ^{-1} = I$.

Worked example: the folium and polar coordinates

- For $x^3 + y^3 = 6xy$ at $(3, 3)$ (check: $54 = 54$): $F_x = 3x^2 - 6y = 9$ and $F_y = 3y^2 - 6x = 9$, so $dy/dx = -1$.
- Polar: $x = r\cos\theta$, so $x_\theta = -r\sin\theta$. But $\theta_x = -\sin\theta/r$, not $1/x_\theta = -1/(r\sin\theta)$.
- At the origin of the folium, $F_x = F_y = 0$: the curve crosses itself, and no single tangent exists.

Common errors

- Writing $u_x = 1/x_u$.
- Forgetting the minus sign in $dy/dx = -F_x/F_y$.
- Using the theorem where $F_y = 0$ or $\det J = 0$.

Common errors (2)

- Evaluating inverse partials at the input point's coordinates instead of matching input and output points.
- Forgetting the absolute value when the Jacobian scales areas.

Key terms

- **Jacobian matrix**: The matrix of first partial derivatives of a map between spaces.
- **local inverse**: An inverse map defined near one input point and its output.

Check yourself

- What is dy/dx on $x^2 + y^2 = 25$ at $(3, 4)$?
- A map has $\det J = 4$ at a point. What is the determinant of the inverse map's Jacobian there?
- Can $x^2 + y^2 = 1$ be written as $y = f(x)$ near $(1, 0)$?