T.11 · Weak compositions, loop invariants and flowchart tracing

Original classroom teaching; not an official form or score predictor

Weak compositions, loop invariants and flowchart tracing

- Count each **weak composition** 弱组合分拆 with stars and bars.
- Handle lower bounds by shifting and upper bounds by inclusion–exclusion.
- Trace loops and flowcharts, counting iterations.
- State and use a loop invariant.

Stars and bars

- Picture n stars and $k-1$ bars in a row; the bars split the stars into k groups.
- Positive integers (each $x_i \geq 1$): $\binom{n-1}{k-1}$.
- Lower bounds: substitute $x_i = u_i + m_i$ to remove each bound $x_i \geq m_i$.
- Upper bounds: count all solutions, then subtract those that break a bound, with inclusion–exclusion.

$$\binom{n+k-1}{k-1}$$

Figure: Stars and bars



7 slots, choose where the 2 bars go:
$\binom{5+3-1}{3-1} = \binom{7}{2} = 21$ solutions
of $x_1 + x_2 + x_3 = 5,\ x_i \geq 0$

- Stars and bars: C(n + k - 1, k - 1).
- Match each part of the diagram to the text.

Tracing algorithms

- Make a table with one column per variable and one row per pass, and update the variables in the order the code does.
- Check the loop condition at the right moment: before the body for "while", after it for "repeat until".
- Count iterations carefully: off-by-one errors are the commonest mistake.

Loop invariants

- True before the first iteration.
- Kept true by every iteration.
- Useful at the end: combined with the stopping condition, it gives the result.

Figure: Loop invariants

Odd-sum loop: $S = m^2$, $B = 2m + 1$ before the next addition

Iterations m	Sum S	Next addend B
0	0	1
1	1	3
2	4	5
3	9	7
4	16	9

- The loop invariant $S = m^2$ holds before each addition
- Match each part of the diagram to the text.

Worked example: bounded solutions

- Put $u = a - 2$: $u + b + c = 6$ has $\binom{8}{2} = 28$ solutions.
- With also $b, c \geq 1$: subtract 1 from each to get a total of 4, so $\binom{6}{2} = 15$.
- So $28 - 15 = 13$ have $b = 0$ or $c = 0$. Check: $b = 0$ gives 7, $c = 0$ gives 7, both zero is counted twice: $7 + 7 - 1 = 13$.

Worked example: an odd-number loop

- Invariant after m passes: $S = m^2$ and $B = 2m + 1$. It holds at $m = 0$, and one pass gives $S = m^2 + 2m + 1 = (m + 1)^2$, $B = 2(m + 1) + 1$.
- The loop stops when $B = 2n + 1$, that is after n passes, so the output is n^2. For $n = 4$: 16.
- The loop ‘while x $>$ 1: x = x // 2‘, from $x = 100$, runs through 50, 25, 12, 6, 3, 1: six iterations.

Common errors

- Using $\binom{n+k-1}{k}$ with the wrong lower index, or mixing up nonnegative and positive solutions.
- Forgetting to shift lower bounds before counting.
- Counting a loop condition check as an extra pass.

Common errors (2)

- Stating an invariant that is true at the end but not kept by each pass.
- Updating two variables in the wrong order when tracing.

Key terms

- **weak composition**: An ordered way of writing a number as a sum of nonnegative integers.
- **loop invariant**: A statement that is true before and after every iteration of a loop.

Check yourself

- How many nonnegative solutions does $x + y + z = 10$ have?
- How many positive solutions does $a + b + c + d = 10$ have?
- What does the odd-number loop output for $n = 7$?