# C.9 · Integrating factors and nonhomogeneous differential equations

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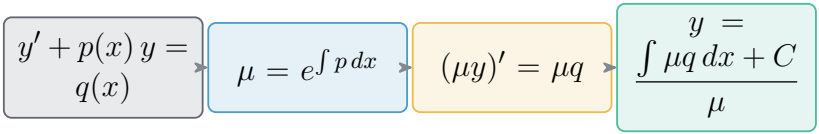
# Integrating factors and nonhomogeneous differential equations

- Solve first-order linear equations with an **integrating factor** 积分因子.
- Write general solutions as homogeneous plus **particular solution** 特解.
- Choose trial solutions by undetermined coefficients, including the resonance case.
- Check solutions and use initial conditions.

# First-order linear equations

- Multiply by $\mu = e^{\int p\,dx}$. The left side becomes $(\mu y)'$.
- Integrate: $\mu y = \int \mu q\,dx + C$, so $y = \dfrac{1}{\mu}\left(\int \mu q\,dx + C\right)$.
- The constant must be included before dividing by $\mu$.

# Figure: First-order linear equations




apply the initial value only after integrating; any antiderivative of $p$ will do

- Integrating factor in four steps.
- Match each part of the diagram to the text.

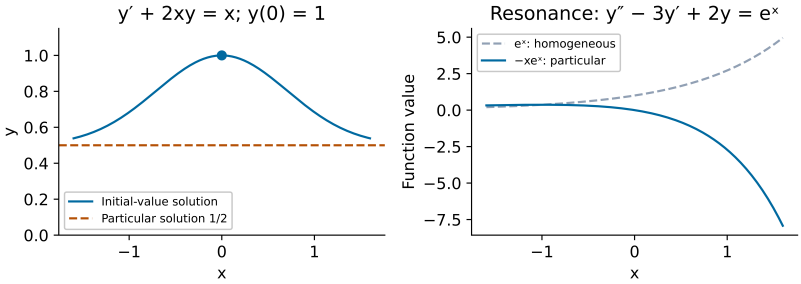
# Superposition

- The general solution is $y = y_h + y_p$: all solutions of $L[y] = 0$, plus one solution of $L[y] = g$.
- If $g = g_1 + g_2$, add particular solutions for each part.
- Apply the initial conditions to the full $y$, not to $y_h$ alone.

# Undetermined coefficients

- $g = e^{kx}$: try $Ae^{kx}$. $g$ a polynomial of degree $n$: try a general polynomial of degree $n$.
- $g = \cos \omega x$ or $\sin \omega x$: try $A \cos \omega x + B \sin \omega x$.
- Resonance: if the trial already solves the homogeneous equation, multiply it by $x$ (by $x^2$ for a double root).

# Figure: Undetermined coefficients




- When the forcing term solves the homogeneous equation, multiply the trial solution by x
- Match each part of the diagram to the text.

# Worked example: an integrating factor

- $\mu = e^{2x}$, so $(e^{2x}y)' = e^{3x}$ and $e^{2x}y = \frac{1}{3}e^{3x} + C$.
- $y = \frac{1}{3}e^{x} + Ce^{-2x}$, and $y(0) = 1$ gives $C = \frac{2}{3}$.
- Check: $y' + 2y = \frac{1}{3}e^{x} - \frac{4}{3}e^{-2x} + \frac{2}{3}e^{x} + \frac{4}{3}e^{-2x} = e^{x}$.
- Similarly $y' + 2xy = x$, $y(0) = 1$ has $\mu = e^{x^2}$ and solution $y = \frac{1}{2} + \frac{1}{2}e^{-x^2}$, which tends to $\frac{1}{2}$.

# Worked example: resonance

- The characteristic roots are 1 and 2, so $e^x$ solves the homogeneous equation and $Ae^x$ fails.
- Try $y = Axe^x$: $y' = A(x+1)e^x$, $y'' = A(x+2)e^x$, and $y'' - 3y' + 2y = A\left[(x+2) - 3(x+1) + 2x\right]e^x = -Ae^x$.
- So $A = -1$, $y_p = -xe^x$, and $y = C_1e^x + C_2e^{2x} - xe^x$.

# Common errors

- Forgetting the constant of integration before dividing by $\mu$.
- Using $e^{p}$ instead of $e^{\int p\,dx}$.
- Using a trial solution that already solves the homogeneous equation.

# Common errors (2)

- Applying initial conditions before adding $y_p$.
- Writing the equation with a coefficient on $y'$ and not dividing by it first.

# Key terms

- **integrating factor**: A multiplier that turns a first-order linear equation into the derivative of a product.
- **particular solution**: One solution of the nonhomogeneous equation, added to the homogeneous solutions.

# Check yourself

- What is the integrating factor for $y' + y/x = 1$, $x > 0$?
- What trial solution suits $y'' + y = \sin x$?
- What is the general solution of $y' - y = 2$?