T.10 · Vector geometry, projections and oriented area

Original classroom teaching; not an official form or score predictor

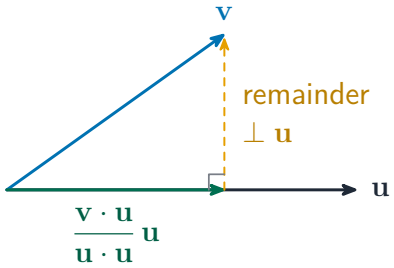
Vector geometry, projections and oriented area

- Use the dot product for angles, **orthogonal** 正交 vectors and projections.
- Use the **cross product** 叉积 for areas, normals and orientation.
- Use the triple product for volumes.
- Write lines and planes, and find distances.

The dot product

- $u \cdot v = 0$ exactly when u and v are orthogonal.
- The scalar component of v along u is $\dfrac{u \cdot v}{|u|}$, a signed number.
- The vector projection is $\operatorname{proj}_u v = \dfrac{u \cdot v}{u \cdot u} u$. The remainder $v - \operatorname{proj}_u v$ is orthogonal to u.

Figure: The dot product



scalar component: $\dfrac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{u}|}$

(a signed number, not a vector)

- Projection and orthogonal remainder.
- Match each part of the diagram to the text.

The cross product

- It is perpendicular to both u and v, with direction given by the right-hand rule.
- $|u \times v| = |u||v| \sin\theta$, the area of the parallelogram on u and v. A triangle has half that area.
- $v \times u = -(u \times v)$: the order sets the orientation.

Volumes

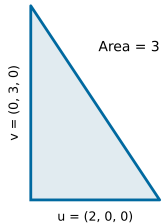
- The triple product $u \cdot (v \times w)$ equals the determinant with rows u, v, w.
- Its absolute value is the volume of the parallelepiped; it is zero exactly when the three vectors lie in one plane.

Lines and planes

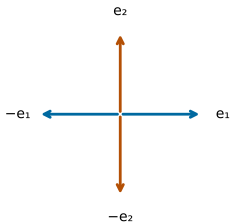
- A line: $r = r_0 + tv$.
- A plane with normal $n = (a, b, c)$: $ax + by + cz = d$.
- Distance from (x_0, y_0, z_0) to that plane: $\dfrac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$.

Figure: Lines and planes

Triangle area = |u × v| / 2 = 3



Two negative products; four zeros



- Triangle area $= |u \times v| \,/\, 2 = 6 \,/\, 2 = 3$
- Match each part of the diagram to the text.

Worked example: projection and area

- $u \cdot v = 14$ and $u \cdot u = 5$, so $\operatorname{proj}_u v = \frac{14}{5}(1,0,2) = (2.8, 0, 5.6)$.
- The remainder is $(-0.8, 3, 0.4)$, and its dot product with u is $-0.8 + 0.8 = 0$.
- $u \times v = (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1) = (0-6,\ 4-6,\ 3-0) = (-6, -2, 3)$, of length $\sqrt{36 + 4 + 9} = 7$, so the triangle O, u, v has area $\frac{7}{2}$.

Worked example: distance and volume

- Distance from $(1,2,3)$ to $x+2y+2z=4$: $\dfrac{|1+4+6-4|}{3} = \dfrac{7}{3}$.
- The parallelepiped on $(1,0,0)$, $(1,2,0)$, $(1,1,3)$ has volume $|\det| = 1 \times (2 \times 3 - 0) = 6$.
- The angle between $(1,1,0)$ and $(0,1,1)$: $\cos\theta = \dfrac{1}{\sqrt{2}\sqrt{2}} = \dfrac{1}{2}$, so $60°$.

Common errors

- Confusing the scalar component with the vector projection.
- Dividing by $|u|$ instead of $u \cdot u$ in the vector projection.
- Reversing the order in a **cross product** and losing the orientation.

Common errors (2)

- Forgetting the factor $\frac{1}{2}$ for a triangle.
- Dropping the absolute value in the distance formula.

Key terms

- **orthogonal**: Having zero dot product.
- **cross product**: The vector $u \times v$, perpendicular to u and v, whose length is the area of the parallelogram they span.

Check yourself

- Are $(1, 2, -1)$ and $(3, -1, 1)$ orthogonal?
- What is $(1, 0, 0) \times (0, 1, 0)$?
- What is the area of the triangle with vertices $(0, 0, 0)$, $(2, 0, 0)$ and $(0, 3, 0)$?