C.8 · Trigonometric phase and parametric curves

Original classroom teaching; not an official form or score predictor

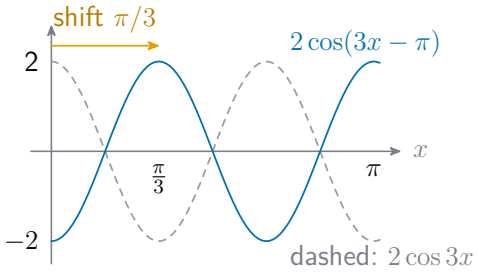
Trigonometric phase and parametric curves

- Read amplitude, period, **phase angle 相位角** and horizontal shift.
- Combine $a\cos x + b\sin x$ into one wave.
- Parametrise curves and find slopes, lengths and areas from a parametrisation.

Sinusoids

- Amplitude $|A|$, period $2\pi/\omega$.
- The phase is φ, but the horizontal shift is φ/ω, to the right. They are equal only when $\omega = 1$.
- A negative coefficient is a phase of π: $-3\cos 2x = 3\cos(2x - \pi)$.

Figure: Sinusoids



- Horizontal shift = phase / omega.
- Read the axes and units first.

Combining waves

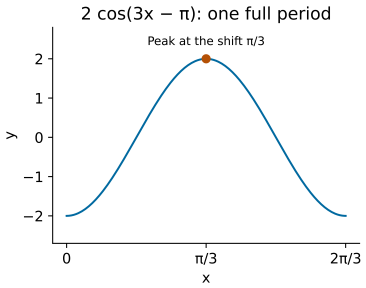
- The maximum is R and the minimum $-R$.
- Use both $\cos\alpha$ and $\sin\alpha$ to place α in the right quadrant; $\tan\alpha = b/a$ alone is ambiguous.

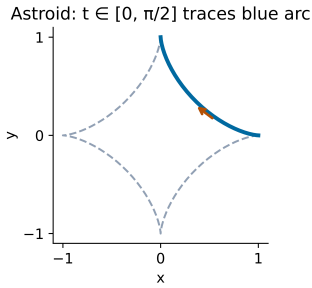
$$a\cos x + b\sin x = R\cos(x-\alpha), \qquad R = \sqrt{a^2+b^2}, \quad \cos\alpha = \frac{a}{R},\ \sin\alpha = \frac{b}{R}$$

Parametric curves

- Circle: $(r\cos t, r\sin t)$. Ellipse: $(a\cos t, b\sin t)$. Both run anticlockwise as t increases.
- Slope: $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$, where $dx/dt \neq 0$.
- Arc length: $\displaystyle\int \sqrt{x'(t)^2 + y'(t)^2}\, dt$.
- Area under a curve: $\displaystyle\int y \frac{dx}{dt}\, dt$, with the limits in the direction of increasing x.

Figure: Parametric curves





- The phase shift $\pi/3$ moves the peak; the period is $2\pi/3$
- Match each part of the diagram to the text.

Worked example: reading a wave

- Amplitude 3, period π.
- $y = 3\cos(2x - \pi)$: phase π, shift $\pi/2$ to the right.
- $3\sin x + 4\cos x = 5\cos(x - \alpha)$ with $\cos\alpha = \frac{4}{5}$, $\sin\alpha = \frac{3}{5}$, so $\alpha = 0.644$. Its maximum is 5.

Worked example: curves

- The ellipse $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$: $x=3\cos t$, $y=2\sin t$, starting at $(3,0)$ and moving anticlockwise.
- One arch of the cycloid ($0\le t\le 2\pi$): $x'=r(1-\cos t)$, $y'=r\sin t$, so the speed is $2r\sin(t/2)$ and the length is $\displaystyle\int_0^{2\pi} 2r\sin\frac{t}{2}\,dt = 8r$.
- The astroid $x=\cos^3 t$, $y=\sin^3 t$ has total length 6.

Common errors

- Reading the phase as the horizontal shift.
- Forgetting that a negative amplitude is a phase of π.
- Placing α in the wrong quadrant from $\tan\alpha$ alone.

Common errors (2)

- Dividing dx/dt by dy/dt for the slope.
- Running the area integral in the wrong direction and losing the sign.

Key terms

- **parametric curve**: A curve whose coordinates are given as functions of a shared parameter.
- **phase angle**: The angle added inside a periodic function's argument, defined up to a full period.

Check yourself

- What is the period of $\sin 4x$?
- Which way, and how far, is $\cos(2x + \pi/2)$ shifted from $\cos 2x$?
- What is the slope of $x = \cos t$, $y = \sin t$ at $t = \pi/4$?