T.9 · Similarity, scaling and conic distance loci

Original classroom teaching; not an official form or score predictor

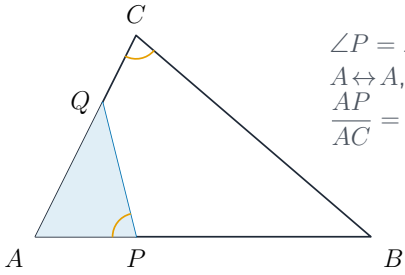
Similarity, scaling and conic distance loci

- Set up similar triangles with the correct vertex correspondence.
- Scale lengths, areas and volumes.
- Use the distance definitions of the conics, each with a **focus** **焦点**.
- Find the axes, foci and eccentricity from an equation.

Similar triangles

- Two triangles are similar if two pairs of angles are equal (AA). Then all corresponding sides are in one ratio.
- Write the correspondence first: in $\triangle APQ \sim \triangle ACB$, $A \leftrightarrow A$, $P \leftrightarrow C$, $Q \leftrightarrow B$, so $\dfrac{AP}{AC} = \dfrac{AQ}{AB} = \dfrac{PQ}{CB}$.
- A line parallel to one side of a triangle cuts off a similar triangle.

Figure: Similar triangles



$\angle P = \angle C$, so
$A \leftrightarrow A$, $P \leftrightarrow C$, $Q \leftrightarrow B$:

$$\frac{AP}{AC} = \frac{AQ}{AB} = \frac{PQ}{CB}$$

- Write the vertex correspondence before the ratios.
- Match each part of the diagram to the text.

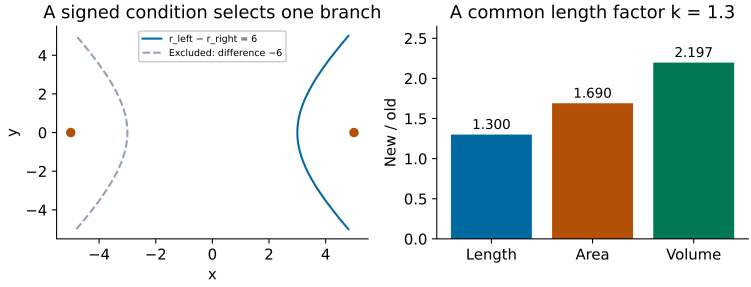
Scaling

- Areas scale by k^2 and volumes by k^3.
- An area ratio of 9 : 25 means a length ratio of 3 : 5 and a volume ratio of 27 : 125.
- An area ratio alone does not prove that two solids are similar.

Conics as loci

- Ellipse: the sum of distances to two foci is $2a$. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $c^2 = a^2 - b^2$, foci $(\pm c, 0)$.
- Hyperbola: the difference of distances is $\pm 2a$. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, $c^2 = a^2 + b^2$. A signed difference picks one branch.
- Parabola: distance to the focus equals distance to the directrix. Focus $(0, p)$ and directrix $y = -p$ give $x^2 = 4py$.
- Eccentricity $e = c/a$: below 1 for an ellipse, 1 for a parabola, above 1 for a hyperbola.

Figure: Conics as loci



- A signed difference picks one branch; a length factor k scales area by k^2 and volume by k^3
- Match each part of the diagram to the text.

Worked example: scale factors

- Similar solids with surface areas $9 : 25$ have lengths $3 : 5$ and volumes $27 : 125$.
- A model at scale $1 : 50$ has $1/2500$ of the real area and $1/125000$ of the volume.
- A 1.3 times enlargement multiplies area by 1.69 and volume by 2.197.

Worked example: finding the conic

- Points with $r_{\text{left}} - r_{\text{right}} = 6$ for foci $(\pm 5, 0)$: $a = 3$, $b^2 = 25 - 9 = 16$, so $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$. Only the right branch, $x \geq 3$, is closer to the right focus.
- Foci $(\pm 3, 0)$ and distance sum 10: $a = 5$, $b^2 = 25 - 9 = 16$, the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$, with $e = 0.6$.
- Focus $(0, 2)$ and directrix $y = -2$: $x^2 = 8y$.

Common errors

- Writing ratios before matching equal angles.
- Scaling area or volume by k instead of k^2 or k^3.
- Using $c^2 = a^2 - b^2$ for a hyperbola.

Common errors (2)

- Giving both branches when the difference has a sign.
- Confusing the focus with the vertex.

Key terms

- **similarity**: Having equal corresponding angles and one common ratio of corresponding lengths.
- **focus**: A fixed point used to define a conic by distances.

Check yourself

- Two similar cones have volumes $8:27$. What is the ratio of their heights?
- What is the eccentricity of $\dfrac{x^2}{16}+\dfrac{y^2}{7}=1$?
- Where is the focus of $y^2=12x$?