

## T.8 · Conditioning, Bayesian inference and sampling error

Original classroom teaching; not an official form or score predictor

# 1. Scope and prerequisites

## Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

## 2. Scope and prerequisites

### Scope and prerequisites

Prerequisites: Probability tables, expectation, variance and independent sampling.

### 3. Scope and prerequisites

#### Scope and prerequisites

Separate conditional, joint and independent-event probabilities Calculate posterior probabilities with a complete base-rate table Use variance and sample size to determine a sample mean standard error

## 4. Scope and prerequisites

### Scope and prerequisites

posterior probability : A probability conditional on the observed evidence after accounting for prior proportions.

## 5. Scope and prerequisites

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standard error : The standard deviation of a statistic across repeated samples.

## 6. Choose and justify a method

### Choose and justify a method

For  $P(B) > 0$ ,  $P(A|B) = P(A \cap B) / P(B)$ : restrict the population to  $B$  before computing the fraction in  $A$ . Independence means  $P(A \cap B) = P(A)P(B)$ , equivalently  $P(A|B) = P(A)$  when the denominator is nonzero. Mutually exclusive events with positive probabilities cannot be independent, because their joint probability is zero. Sampling without replacement usually changes later probabilities; a fixed denominator for successive draws would describe a different experiment.

## 7. Choose and justify a method

### Choose and justify a method

Build joint probabilities by multiplying a prior category proportion by the relevant conditional probability. If  $D$  is the condition and  $+$  a positive result,  $P(D \cap +) = P(D)P(+|D)$ . The total positive probability is  $P(D)P(+|D) + P(\text{not } D)P(+|\text{not } D)$ , because these are disjoint and cover every positive result. Bayes divides the first joint probability by this total. Sensitivity is  $P(+|D)$ , while specificity is  $P(-|\text{not } D)$ ; the false-positive rate is one minus specificity.

## 8. Choose and justify a method

### Choose and justify a method

For a hypothetical test with prevalence 2%, sensitivity 90% and specificity 95%, use a population of 10,000 for clarity. Of 200 people with the condition, 180 test positive. Of 9,800 without it, 490 test positive. Therefore 180 of 670 positive results have the condition: posterior  $18/67$ , about 26.9%. Reversing the conditional would give 90%, answering a different question. This is a mathematical model, not a recommendation about clinical decisions.

## 9. Choose and justify a method

### Choose and justify a method

For independent identically distributed observations with finite variance  $\sigma^2$ , the sample mean has expectation  $\mu$  and variance  $\sigma^2/n$ ; its standard error is  $\sigma/\sqrt{n}$ . This is spread of repeated sample means, not the spread of individual observations. Quadrupling  $n$  halves the standard error, rather than quartering it. Normal population data give an exactly normal mean; otherwise a central-limit approximation needs adequate conditions and sample size. Correlation invalidates the simple independent variance calculation.

## 10. Worked reasoning

### Worked reasoning

Let  $P(A)=0.4$ ,  $P(B)=0.5$  and  $P(A \cap B)=0.3$ . Then  $P(A|B)=0.6$ , so  $A$  and  $B$  are not independent, since  $0.3 \neq 0.4 \cdot 0.5$ . If a variable has standard deviation 12, an independent sample of size 36 has mean standard error 2. Increasing the sample size to 144 gives 1; the individual-observation standard deviation stays 12.

## Figure 11: Worked reasoning

Condition on positives: 80 / 125, not 80 / 100

| Expected group | Positive | Negative | Total |
|----------------|----------|----------|-------|
| Defective      | 80       | 20       | 100   |
| Not defective  | 45       | 855      | 900   |
| Total          | 125      | 875      | 1000  |

Check the labelled values and conditions.

## 12. Conditions and counterexamples

### Conditions and counterexamples

Do not reverse  $P(+|D)$  into  $P(D|+)$ . Include false positives in the posterior denominator. Standard deviation of individual observations and standard error of their mean have different sample-size behaviour.

## 13. Try independently

### Try independently

In a stated model, 10% of devices have a defect. A test is positive for 80% of defective and 5% of nondefective devices. Find the probability of a defect given a positive result using 1000 expected devices.

## 14. Reasoning and check

### Reasoning and check

Expected counts are 100 defective and 900 nondefective. Positive counts are  $100(0.8) = 80$  and  $900(0.05) = 45$ , respectively. The conditioned group contains 125 positives, hence  $P(D | +) = \frac{80}{80+45} = \frac{16}{25} = 0.64$ . Negative counts 20 and 855 check both row totals. The sensitivity 0.8 is  $P(+ | D)$ , not the requested reversed condition. These are illustrative model rates, not measured performance claims.

## 15. Try independently

### Try independently

Independent observations have variance 25. Find the standard error of their mean for  $n=100$ . If instead every distinct pair has covariance 1, find the mean's variance. Does quadrupling  $n$  necessarily halve the standard error in that second model?

## 16. Reasoning and check

### Reasoning and check

Independence gives  $\text{Var}(\bar{X}) = 25/n$  and at 100 standard error  $5/10 = 0.5$ . With covariance one,  $\text{Var}(\bar{X}) = \frac{25n+n(n-1)}{n^2} = 1 + \frac{24}{n}$ . At 100 it is 1.24. The standard error tends to one as  $n$  grows, rather than to zero; quadrupling  $n$  need not halve it. Covariance terms may be dropped only under justified uncorrelatedness, with independence a sufficient condition.

## Check yourself

### Reconstruct the method without looking back

Independent observations have variance 25. Find the standard error of their mean for  $n=100$ . If instead every distinct pair has covariance 1, find the mean's variance. Does quadrupling  $n$  necessarily halve the standard error in that second model?