T.7 · Set images, equivalence relations and logical negation

Original classroom teaching; not an official form or score predictor

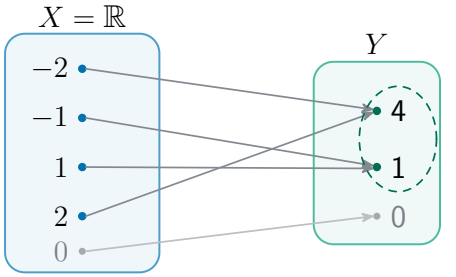
Set images, equivalence relations and logical negation

- Compute images and **preimages** 原像 and know which operations they preserve.
- Check the three properties of an **equivalence relation** 等价关系 and find its classes.
- Negate statements with quantifiers correctly.
- Separate the converse from the contrapositive.

Images and preimages

- A preimage needs no inverse function: for $f(x) = x^2$, $f^{-1}(\{1, 4\}) = \{-2, -1, 1, 2\}$.
- **Preimages** preserve everything: unions, intersections and complements.
- Images preserve unions, but $f(A \cap B) \subseteq f(A) \cap f(B)$ can be strict. Equality for all sets holds exactly when f is injective.

Figure: Images and preimages



$f^{-1}(C) = \{-2, -1, 1, 2\}$
although f has no inverse function

- A preimage needs no inverse function.
- Match each part of the diagram to the text.

Equivalence relations

- Reflexive: $a \sim a$. Symmetric: $a \sim b$ implies $b \sim a$. Transitive: $a \sim b$ and $b \sim c$ imply $a \sim c$.
- Its classes partition the set, and every partition comes from an equivalence relation.
- $a \sim b$ when $3 \mid (a - b)$, on $\mathbb{Z}$, has three classes: the residues 0, 1 and 2 mod 3.
- $\leq$ is reflexive and transitive but not symmetric, so it is not an equivalence relation.

Negation

- $\neg(\forall x\, P(x))$ is $\exists x \neg P(x)$; $\neg(\exists x\, P(x))$ is $\forall x \neg P(x)$.
- $\neg(P \wedge Q)$ is $\neg P \vee \neg Q$; $\neg(P \Rightarrow Q)$ is $P \wedge \neg Q$.
- Negate from the outside in, flipping each quantifier and keeping its order.

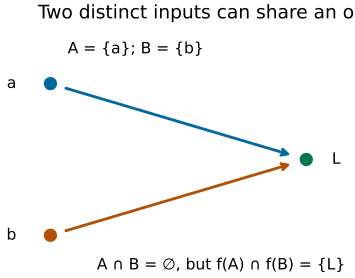
Conditionals

- $P \Rightarrow Q$ is equivalent to its contrapositive $\neg Q \Rightarrow \neg P$.
- The converse $Q \Rightarrow P$ and the inverse $\neg P \Rightarrow \neg Q$ are different statements.
- "Differentiable implies continuous" is true; its converse is false ($|x|$).

Worked example: images of intersections

- If $y = f(x)$ with $x \in A \cap B$, then $y \in f(A)$ and $y \in f(B)$.
- Strict: let $f(a) = f(b) = c$ with $a \neq b$, $A = \{a\}$, $B = \{b\}$. Then $f(A \cap B) = \varnothing$ but $f(A) \cap f(B) = \{c\}$.
- If f is injective and $y = f(a) = f(b)$ with $a \in A$, $b \in B$, then $a = b \in A \cap B$, so equality holds.

Figure: Worked example: images of intersections



- f(A ∩ B) can be empty while f(A) ∩ f(B) is not: images do not preserve intersections
- Match each part of the diagram to the text.

Worked example: negating continuity

- Negation: $\exists \varepsilon > 0 \ \forall \delta > 0 \ \exists x \ (|x - a| < \delta \wedge |f(x) - f(a)| \geq \varepsilon)$.
- In words: some tolerance cannot be met, however close you look.
- "Every student passed some test" negates to "some student failed every test".

Common errors

- Assuming images preserve intersections.
- Thinking a preimage requires an inverse function.
- Negating "for all" as "for none".

Common errors (2)

- Treating the converse as equivalent to the statement.
- Calling a relation an equivalence after checking only two properties.

Key terms

- **preimages**: Sets of all inputs whose outputs lie in a given set.
- **equivalence relation**: A relation that is reflexive, symmetric and transitive.

Check yourself

- For $f(x) = x^2$, what is $f^{-1}([4, 9])$?
- How many equivalence relations are there on a set with 3 elements?
- What is the negation of "$P \Rightarrow Q$"?