T.6 · Functions, inverse branches and composition

Original classroom teaching; not an official form or score predictor

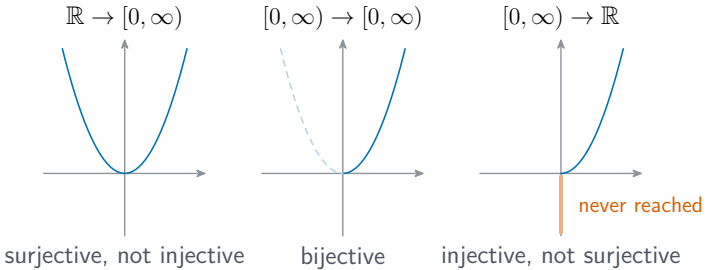
Functions, inverse branches and composition

- Decide whether a function is injective, surjective or a **bijection** 双射, with stated domain and codomain.
- Restrict a function to a branch and find its inverse.
- Use inverse trigonometric functions with their correct ranges.
- Reason about compositions and **involutions** 对合.

Injective, surjective, bijective

- Injective (one-to-one): different inputs give different outputs.
- Surjective (onto): every element of the codomain is an output.
- Bijective: both. Only a **bijection** has an inverse function.
- The same rule x^2 is surjective but not injective from $\mathbb{R}$ to $[0, \infty)$, bijective from $[0, \infty)$ to $[0, \infty)$, and injective but not surjective from $[0, \infty)$ to $\mathbb{R}$.

Figure: Injective, surjective, bijective

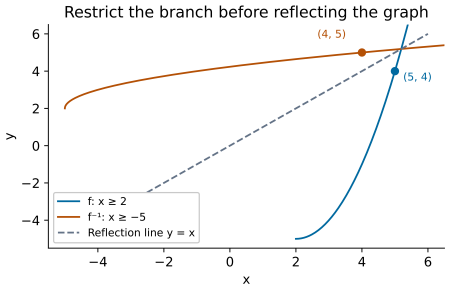


- Same rule, different domain and codomain.
- Read the axes and units first.

Inverse branches

- $\sqrt{y}$ is the inverse of x^2 on $x \geq 0$.
- $\arcsin$ has range $[-\pi/2, \pi/2]$, $\arccos$ has $[0, \pi]$, $\arctan$ has $(-\pi/2, \pi/2)$.
- $\arcsin(\sin x) = x$ only when x is in $[-\pi/2, \pi/2]$; $\sin(\arcsin y) = y$ for every y in $[-1, 1]$.
- The graph of f^{-1} is the reflection of the graph of f in $y = x$.

Figure: Inverse branches



- Restrict to one branch before reflecting: (5, 4) on f becomes (4, 5) on f^{-1}
- Match each part of the diagram to the text.

Composition

- $(g \circ f)(x) = g(f(x))$: apply f first. Usually $g \circ f \neq f \circ g$.
- $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
- If $g \circ f$ is injective, then f is injective. If $g \circ f$ is surjective, then g is surjective.
- An involution is its own inverse: $f(f(x)) = x$. Examples: $1/x$ on $x \neq 0$, $a - x$, and reflection in a line.

Worked example: a branch

- The image is $[2, \infty)$.
- Solving $y = (x-1)^2 + 2$ with $x \geq 1$ gives $x = 1 + \sqrt{y-2}$, so $f^{-1} : [2, \infty) \to [1, \infty)$, $f^{-1}(y) = 1 + \sqrt{y-2}$.
- Check: $f^{-1}(f(x)) = 1 + |x-1| = x$ because $x \geq 1$; $f(f^{-1}(y)) = y$.

Worked example: compositions

- $\arcsin\left(\sin\frac{5\pi}{6}\right) = \arcsin\frac{1}{2} = \frac{\pi}{6}$, not $\frac{5\pi}{6}$.
- $f(x) = \dfrac{x+1}{x-1}$: $f(f(x)) = \dfrac{2x/(x-1)}{2/(x-1)} = x$, so f is an involution on $x \neq 1$.
- If $f(x) = 2x+1$ and $g(x) = x^3$, then $(g \circ f)^{-1}(y) = \dfrac{\sqrt[3]{y}-1}{2}$.

Common errors

- Calling a function bijective without stating the codomain.
- Taking both square-root branches as one inverse.
- Using $\arcsin(\sin x) = x$ outside $[-\pi/2, \pi/2]$.

Common errors (2)

- Reversing the order in $(g \circ f)^{-1}$.
- Assuming composition is commutative.

Key terms

- **bijection**: A function that is both injective and surjective between its stated sets.
- **involutions**: Functions that are their own inverses: $f(f(x)) = x$.

Check yourself

- Is e^x a **bijection** from $\mathbb{R}$ to $\mathbb{R}$?
- What is $\arccos(\cos(-\pi/3))$?
- What is the inverse of $f(x) = \dfrac{2x+1}{x-3}$?