

T.6 · Functions, inverse branches and composition

Original classroom teaching; not an official form or score predictor

1. Scope and prerequisites

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

2. Scope and prerequisites

Scope and prerequisites

Prerequisites: Function domains, quadratic equations and composition.

3. Scope and prerequisites

Scope and prerequisites

Distinguish injectivity and surjectivity using the stated domain and codomain Choose and verify an inverse branch by composing in both directions Trace repeated function composition while preserving the domain

4. Scope and prerequisites

Scope and prerequisites

bijection : A function that is both injective and surjective between its specified sets.

5. Scope and prerequisites

Scope and prerequisites

involution : A function whose composition with itself is the identity on its domain.

6. Choose and justify a method

Choose and justify a method

A function assigns exactly one output to each allowed input. Injective means equal outputs force equal inputs; surjective means every element of the stated codomain is reached. The map $x \mapsto x^2$ from $\mathbb{R}$ to $[0, \infty)$ is surjective but not injective. From $[0, \infty)$ to $[0, \infty)$ it is both, hence bijective. From $[0, \infty)$ to $\mathbb{R}$ it remains injective but fails surjectivity. Keep domain, image and codomain separate when testing each claim.

7. Choose and justify a method

Choose and justify a method

An inverse function reverses a bijection. Solve $y=f(x)$ for x , then use the original domain to choose a branch. For $f(x)=(x-2)^2-5$ on $x \geq 2$, the inverse is $2+\sqrt{y+5}$ on $y \geq -5$. The minus branch would return inputs outside the chosen domain. Check $f^{-1}(f(x))=x$ for allowed x and $f(f^{-1}(y))=y$ for allowed y ; one unchecked composition can conceal a domain error.

8. Choose and justify a method

Choose and justify a method

The inverse graph reflects the original graph across $y=x$; it does not take reciprocals of output values. A self-inverse function, or involution, satisfies $f(f(x))=x$ wherever the composition is defined. Both $-x$ on $\mathbb{R}$ and $1/x$ on $\mathbb{R}$ excluding zero are involutions. A strictly increasing involution on an interval must be the identity: if $f(x)>x$, increasingness gives $f(f(x))>f(x)>x$, and the analogous argument rules out $f(x)<x$.

9. Choose and justify a method

Choose and justify a method

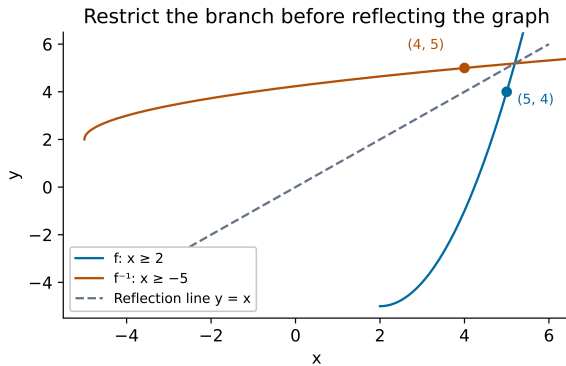
Iteration f^n means repeated composition, not the power $(f(x))^n$. Calculate the first few compositions and check their domains before looking for a period. For $f(x)=1/(1-x)$ on $\mathbb{R}$ excluding 0 and 1, $f^2(x)=(x-1)/x$ and $f^3(x)=x$. The image stays in the same allowed domain. Therefore reduce an iteration count modulo three. A displayed formula equal to x after cancellation does not restore forbidden inputs.

10. Worked reasoning

Worked reasoning

For $f(x)=(x-2)^2-5$ with $x \geq 2$, $f(5)=4$ and $f^{-1}(4)=2+\sqrt{9}=5$. The unrestricted quadratic would have two inputs, -1 and 5 , giving output 4 . For $g(x)=1/(1-x)$, the orbit $3 \rightarrow -1/2 \rightarrow 2/3 \rightarrow 3$ has period three, so $g^8(3)=2/3$; the count concerns compositions, not eighth powers.

Figure 11: Worked reasoning



Check the labelled values and conditions.

12. Conditions and counterexamples

Conditions and counterexamples

Changing a codomain changes surjectivity. Reflection across $y=x$ is an inverse graph; reciprocating y -values is a different operation. Do not cancel away excluded inputs.

13. Try independently

Try independently

Restrict $f(x) = (x - 1)^2 + 2$ to $x \geq 1$. State its image and inverse, including domains. Verify both inverse compositions.

14. Reasoning and check

Reasoning and check

The image is $[2, \infty)$. Solving $y = (x - 1)^2 + 2$ with $x \geq 1$ selects $x = 1 + \sqrt{y - 2}$. Thus $f^{-1} : [2, \infty) \rightarrow [1, \infty)$ has that formula. For allowed x , $1 + \sqrt{(x - 1)^2} = x$ because $x - 1 \geq 0$. For allowed y , $(\sqrt{y - 2})^2 + 2 = y$. The negative square-root branch belongs to a different restriction, not a second inverse on this domain.

15. Try independently

Try independently

Let $h(x) = 1/(1 - x)$. Determine the domain of $h \circ h \circ h$ and its formula. May the simplified expression be extended to the excluded points without changing the original composite?

16. Reasoning and check

Reasoning and check

The first stage requires $x \neq 1$. The second requires $h(x) \neq 1$, which excludes $x=0$. For $x \neq 0, 1$, $h^2(x) = (x-1)/x$. This never equals one, so the third stage adds no exclusion. Then $h^3(x) = x$ on $\mathbb{R} \setminus \{0, 1\}$. The identity formula is defined more widely, but adding 0 or 1 defines an extension, not the original composite. Each intermediate input must be permitted before cancellation.

Check yourself

Reconstruct the method without looking back

Let $h(x) = 1/(1 - x)$. Determine the domain of $h \circ h \circ h$ and its formula. May the simplified expression be extended to the excluded points without changing the original composite?