C.7 · Coordinate changes and vector integral theorems

Original classroom teaching; not an official form or score predictor

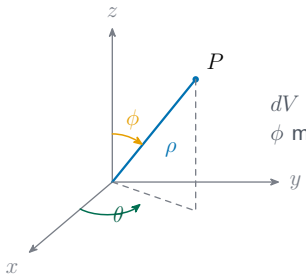
Coordinate changes and vector integral theorems

- Integrate in polar, cylindrical and spherical coordinates with the right Jacobian.
- Compute the **divergence** 散度 and curl of a field.
- Decide whether a field is a **conservative field** 保守场.
- Apply Green's, Stokes' and the **divergence** theorems with the right orientation.

Coordinate changes

- Polar: $dA = r\,dr\,d\theta$. Cylindrical: $dV = r\,dr\,d\theta\,dz$.
- Spherical, with ϕ measured from the positive z-axis: $dV = \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$.
- A ball of radius R: $\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \frac{4}{3}\pi R^3$.

Figure: Coordinate changes



$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

ϕ measured from the $+z$ axis

- Spherical coordinates and their Jacobian.
- Read the axes and units first.

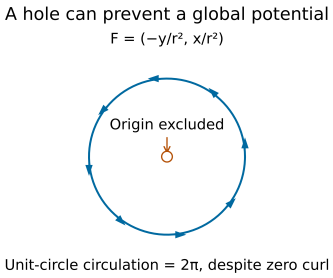
Divergence and curl

- $\operatorname{div} F = P_x + Q_y + R_z$, the outflow per unit volume.
- $\operatorname{curl} F = \nabla \times F$, the circulation per unit area.
- $\operatorname{curl}(\nabla f) = 0$ and $\operatorname{div}(\operatorname{curl} F) = 0$ for smooth fields.

Conservative fields

- A **conservative field** has zero curl.
- Zero curl implies conservative only on a simply connected region, one with no holes.
- $F = \left(\dfrac{-y}{x^2 + y^2}, \dfrac{x}{x^2 + y^2} \right)$ has zero curl away from the origin, but its circulation around the unit circle is 2π.

Figure: Conservative fields



- Zero curl but circulation 2π: the hole at the origin prevents a global potential
- Match each part of the diagram to the text.

The integral theorems

- Green's theorem is the flat case of Stokes' theorem: $\oint P\,dx + Q\,dy = \iint (Q_x - P_y)\,dA$.
- Orientation: the boundary curve runs anticlockwise when seen from the side the normal points to; closed surfaces use the outward normal.
- The field must be smooth on the whole region, holes included.

$$\oint_{\partial S} F \cdot dr = \iint_S (\nabla \times F) \cdot dS, \qquad \iint_{\partial V} F \cdot dS = \iiint_V \nabla \cdot F\,dV$$

Worked example: an annulus and a sphere

- $\iint (x^2 + y^2)\, dA$ over $1 \le x^2 + y^2 \le 4$: $\displaystyle\int_0^{2\pi} \int_1^2 r^3 \, dr \, d\theta = 2\pi \times \frac{16 - 1}{4} = \frac{15\pi}{2}.$
- Outward flux of $F = (x, y, z)$ through the unit sphere: $\operatorname{div} F = 3$, so the flux is $3 \times \frac{4}{3}\pi = 4\pi$.
- Check directly: on the sphere $F \cdot n = 1$, and the area is 4π.

Worked example: a field with a hole

- $Q_x - P_y = 0$ wherever $(x, y) \neq 0$.
- On $x = \cos t$, $y = \sin t$: $F \cdot dr = (\sin^2 t + \cos^2 t)\, dt$, so the circulation is 2π.
- Green's theorem cannot be used on the disc, because F is undefined at the origin. On a region that does not surround the origin, the circulation is 0.

Common errors

- Writing $dV = \rho \sin \phi$ or forgetting the Jacobian.
- Assuming zero curl means conservative on a region with a hole.
- Using an inward normal without changing the sign.

Common errors (2)

- Applying a theorem where the field is not defined.
- Mixing up the roles of ϕ and θ.

Key terms

- **divergence**: $P_x + Q_y + R_z$, a field's net outflow per unit volume.
- **conservative field**: A field that is the gradient of a potential, so its line integrals depend only on the endpoints.

Check yourself

- What is the **divergence** of (x^2, y^2, z^2)?
- What is the outward flux of $(0, 0, z)$ through the unit sphere?
- Is $(2xy, x^2)$ conservative on the plane? If so, give a potential.