C.6 · Multivariable extrema and constrained optimisation

Original classroom teaching; not an official form or score predictor

Multivariable extrema and constrained optimisation

- Find critical points of functions of several variables.
- Classify them with the **Hessian 海森矩阵** and know when the test fails.
- Find global extrema on closed bounded regions.
- Use a **Lagrange multiplier 拉格朗日乘子** for constrained problems.

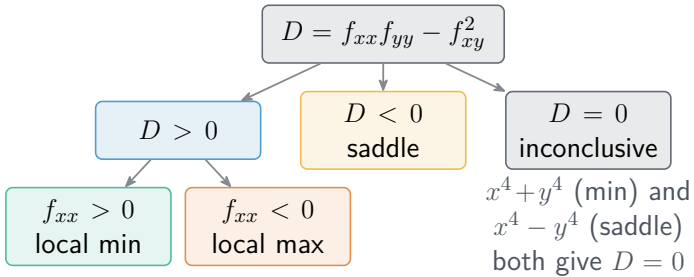
Critical points

- Local extrema of a differentiable function can occur only at critical points.
- Not every critical point is an extremum: $x^2 - y^2$ has a saddle at the origin.

The second-derivative test

- $D > 0$ and $f_{xx} > 0$: local minimum. $D > 0$ and $f_{xx} < 0$: local maximum.
- $D < 0$: saddle point.
- $D = 0$: the test says nothing. $x^4 + y^4$ (a minimum) and $x^4 - y^4$ (a saddle) both have $D = 0$ at the origin.

Figure: The second-derivative test



- The second-derivative test, including its blind spot.
- Match each part of the diagram to the text.

Global extrema

- Check the critical points inside the region.
- Check the boundary, by parametrising it or with Lagrange multipliers.
- Compare all the candidate values.

Lagrange multipliers

- This works where $\nabla g \neq 0$; points where $\nabla g = 0$ must be checked separately.
- The method gives candidates; compare their values to decide which is the maximum or minimum.
- With two constraints, use $\nabla f = \lambda \nabla g + \mu \nabla h$.

$$\nabla f = \lambda \nabla g, \qquad g = c$$

Worked example: classifying

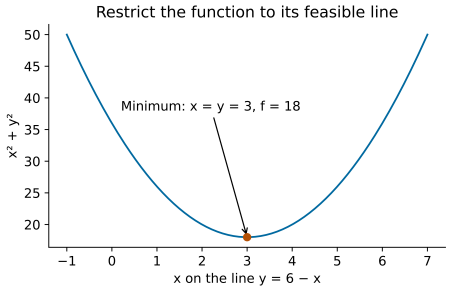
- $f = x^2 - y^2$: the **Hessian** is $\operatorname{diag}(2, -2)$, $D = -4 < 0$, so the origin is a saddle.
- $g = x^4 + y^4$: the Hessian at 0 is zero, so the test is inconclusive. But $g \geq 0$ with equality only at the origin, so it is a strict global minimum.
- $h = x^2 + xy + y^2$: $D = 2 \times 2 - 1 = 3 > 0$ and $h_{xx} = 2 > 0$, a minimum.

Worked example: constraints

- Minimise $x^2 + y^2$ on $x + y = 6$: $(2x, 2y) = \lambda(1, 1)$ gives $x = y = 3$, value 18. Substituting $y = 6 - x$ gives the same result.
- Maximise xy on $x^2 + y^2 = 8$: $(y, x) = \lambda(2x, 2y)$ gives $y^2 = x^2$. With $x = y = 2$, $xy = 4$; with $x = -y$, $xy = -4$. So the maximum is 4 and the minimum -4.
- A box with no lid and volume 32 has least surface area when it is $4 \times 4 \times 2$.

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Figure: Worked example: constraints



- Substitute the constraint first: the minimum is at $x = y = 3$, where $x^2 + y^2 = 18$
- Read the axes and units first.

Common errors

- Treating $D = 0$ as a saddle or a minimum.
- Forgetting the boundary when finding global extrema.
- Reading every Lagrange candidate as a maximum.

Common errors (2)

- Ignoring points where $\nabla g = 0$.
- Using f_{xx} alone to classify a point.

Key terms

- **Hessian**: The matrix of second partial derivatives, used to study the shape of a function near a point.
- **Lagrange multiplier**: The scalar λ in $\nabla f = \lambda \nabla g$ at a constrained extremum.

Check yourself

- Classify the origin for $f = xy$.
- What is the maximum of $x + y$ on the unit circle?
- At a critical point, $f_{xx} = 1$, $f_{yy} = 4$, $f_{xy} = 2$. What does the test say?