C.5 · Improper integrals and geometric applications

Original classroom teaching; not an official form or score predictor

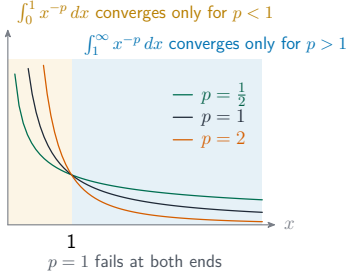
Improper integrals and geometric applications

- Decide whether improper integrals converge, using p-tests and comparison.
- Separate an ordinary improper integral from a **principal value** 主值.
- Find areas, volumes by discs and shells, arc lengths and surface areas.

Improper integrals

- $\int_1^\infty x^{-p}\,dx$ converges exactly when $p > 1$.
- $\int_0^1 x^{-p}\,dx$ converges exactly when $p < 1$.
- If the integral is improper at both ends, split it: both parts must converge separately.
- Comparison: if $0 \le f \le g$ and $\int g$ converges, so does $\int f$. Near infinity, $e^{-x^2} \le e^{-x}$ for $x \ge 1$.

Figure: Improper integrals



- Which p-integrals converge, and where.
- Read the axes and units first.

Principal values

- $\displaystyle\int_{-1}^{1} \frac{dx}{x}$ diverges: each half diverges on its own.
- With symmetric cut-offs, $\displaystyle\lim_{\varepsilon\to 0}\left(\int_{-1}^{-\varepsilon} + \int_{\varepsilon}^{1}\right)\frac{dx}{x} = 0.$ This is the **principal value**, not the value of the ordinary integral.

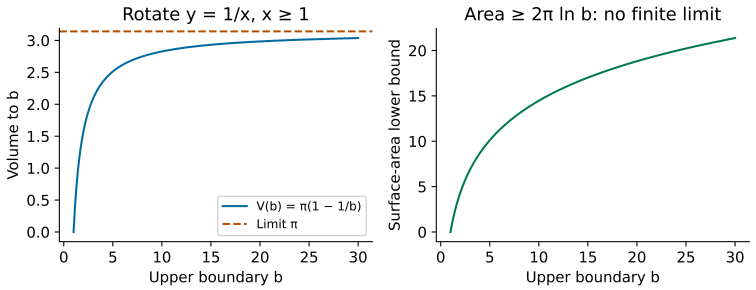
Geometry with integrals

- Area between curves: $\int (\text{top} - \text{bottom})\, dx$.
- Volume by discs, rotating about the x-axis: $\pi \int f(x)^2\, dx$.
- Volume by shells, rotating about the y-axis: $2\pi \int x f(x)\, dx$.
- Arc length: $\int \sqrt{1 + f'(x)^2}\, dx$. Surface area about the x-axis: $2\pi \int f\sqrt{1 + f'^2}\, dx$.

Worked example: Gabriel's horn

- Volume: $\pi \displaystyle\int_1^b \frac{dx}{x^2} = \pi\left(1 - \frac{1}{b}\right) \to \pi.$
- Surface area: $2\pi \displaystyle\int_1^b \frac{1}{x}\sqrt{1 + x^{-4}}\,dx \geq 2\pi \int_1^b \frac{dx}{x} = 2\pi \ln b \to \infty.$
- $\displaystyle\int_0^\infty x^{-p}\,dx$ diverges for every p: near 0 it needs $p < 1$, near infinity $p > 1$.

Figure: Worked example: Gabriel's horn



- Gabriel's horn: finite volume π, infinite surface area
- Match each part of the diagram to the text.

Worked example: shells and length

- Rotate the region under $y = x^2$, $0 \le x \le 1$, about the y-axis: $V = 2\pi \displaystyle\int_0^1 x \cdot x^2 \, dx = \frac{\pi}{2}$.
- Arc length of $y = x^{3/2}$ from 0 to 4: $\displaystyle\int_0^4 \sqrt{1 + \tfrac{9}{4}x}\, dx = \frac{8}{27}\left[\left(1 + \tfrac{9}{4}x\right)^{3/2}\right]_0^4 = \frac{8}{27}\left(10^{3/2} - 1\right) = 9.07.$
- $\displaystyle\int_0^\infty x^n e^{-x}\, dx = n!$, so $\displaystyle\int_0^\infty x^3 e^{-x}\, dx = 6.$

Common errors

- Evaluating an integral across a singularity as if it were ordinary.
- Reporting a **principal value** as the integral's value.
- Using $\pi \int f^2$ when rotating about the other axis.

Common errors (2)

- Forgetting to split an integral that is improper at both ends.
- Thinking a finite volume implies a finite surface area.

Key terms

- **improper integral**: An integral defined as a limit, because a limit of integration or the integrand is infinite.
- **principal value**: A limit with symmetric cut-offs around a singularity, which can exist when the ordinary improper integral diverges.

Check yourself

- Does $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$ converge, and to what?
- Does $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converge?
- Rotate $y = \sqrt{x}$, $0 \le x \le 4$, about the x-axis. What is the volume?