# C.4 · Taylor expansions and power-series endpoints

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# Taylor expansions and power-series endpoints

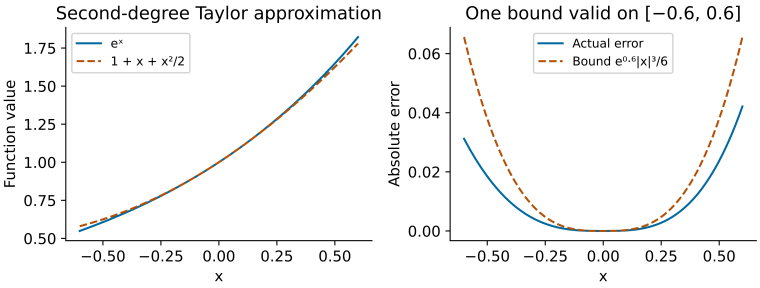
- Find Taylor polynomials and bound the **remainder** 余项.
- Find the **radius of convergence** 收敛半径 and test the endpoints.
- Differentiate and integrate power series term by term.
- Use standard series to find sums and coefficients.

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

# Taylor polynomials

- Lagrange **remainder**: $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some $c$ between $a$ and $x$.
- To bound the error, bound $|f^{(n+1)}|$ on the whole interval.

# Figure: Taylor polynomials

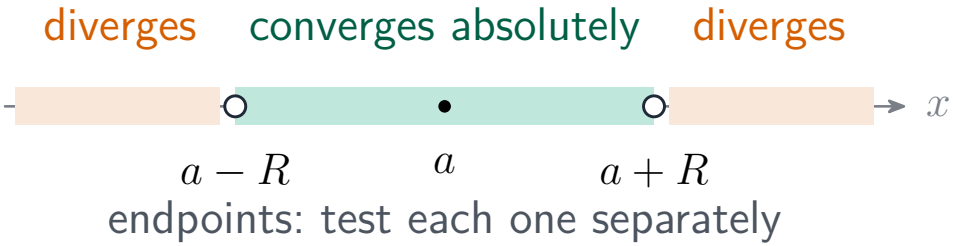



- One Lagrange bound covers the whole interval $[-0.6, 0.6]$
- Match each part of the diagram to the text.

# Radius of convergence

- Find $R$ with the ratio test: $R = \lim |c_n / c_{n+1}|$ when this limit exists.
- At $|x - a| = R$ anything can happen. Test each endpoint separately.
- $\sum n!\, x^n$ has $R = 0$; $\sum x^n / n!$ has $R = \infty$.

## Figure: Radius of convergence




- Inside R, outside R, and the endpoints.
- Match each part of the diagram to the text.

# Term-by-term operations

- Inside the radius, a power series can be differentiated and integrated term by term. The radius stays the same.
- The endpoints can change: differentiating can lose an endpoint, and integrating can gain one.

# Standard series

- $e^x = \sum x^n/n!$; $\sin x = x - x^3/3! + \cdots$; $\cos x = 1 - x^2/2! + \cdots$, all for every $x$.
- $\dfrac{1}{1-x} = \sum x^n$ for $|x| < 1$.
- $\ln(1+x) = x - x^2/2 + x^3/3 - \cdots$ for $-1 < x \le 1$.
- $\arctan x = x - x^3/3 + x^5/5 - \cdots$ for $-1 \le x \le 1$.

# Worked example: endpoints

- The ratio test gives $R = 1$.
- At $x = 1$: the harmonic series diverges. At $x = -1$: the alternating harmonic series converges. Interval: $[-1, 1)$.
- The derivative is $\sum x^{n-1} = 1/(1-x)$, whose terms do not tend to 0 at $\pm 1$. Interval: $(-1, 1)$. The derivative lost the endpoint $-1$.

# Worked example: an error bound

- $P_2(0.1) = 1 + 0.1 + 0.005 = 1.105$.
- On $[0, 0.1]$, $f''' = e^x < 2$, so $|R_2| \le 2(0.1)^3/6 = 1/3000 \approx 0.00033$.
- The true value is $1.10517$, an error of 0.00017, within the bound. For Madhava's series, the alternating test bounds the error after $n$ terms by the next term, $1/(2n+1)$: a thousand terms give only about three digits of $\pi/4$.

# Common errors

- Deciding the endpoints with the ratio test, which gives 1 there.
- Assuming the derivative series has the same interval, endpoints included.
- Bounding the **remainder** at one point instead of over the interval.

# Common errors (2)

- Forgetting the factorial in Taylor coefficients.
- Using $\ln(1+x)$ at $x=-1$.

# Key terms

- **remainder**: The difference between a function and its Taylor polynomial.
- **radius of convergence**: The distance from the centre within which a power series converges absolutely.

# Check yourself

- What is the coefficient of $x^4$ in the Maclaurin series of $\cos x$?
- What is the interval of convergence of $\sum x^n/n^2$?
- What is the radius of convergence of $\sum 2^n x^n$?