

C.4 · Taylor expansions and power-series endpoints

Original classroom teaching; not an official form or score predictor

1. Scope and prerequisites

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

2. Scope and prerequisites

Scope and prerequisites

Prerequisites: Derivatives, geometric series and convergence tests.

3. Scope and prerequisites

Scope and prerequisites

Construct Taylor polynomials and control approximation error
Determine radii of convergence and test endpoints separately
Differentiate and integrate power series within their interval of convergence

4. Scope and prerequisites

Scope and prerequisites

radius of convergence : The distance from the series centre inside which a power series converges absolutely.

5. Scope and prerequisites

Scope and prerequisites

remainder : The difference between a function and its finite approximation.

6. Choose and justify a method

Choose and justify a method

The Taylor polynomial of degree m at a is the sum of $f^{(k)}(a)(x-a)^k/k!$ for k from zero to m . The factorial belongs to each coefficient. If the next derivative is bounded in magnitude by M between a and x , the Lagrange remainder has magnitude at most $M|x-a|^{m+1}/(m+1)!$. Smoothness alone does not guarantee that the infinite Taylor series equals the function everywhere.

7. Choose and justify a method

Choose and justify a method

A power series $\sum c_n(x-a)^n$ converges absolutely inside its radius R and diverges outside it. Ratio or root tests usually determine R , with possible values zero and infinity. At $x=a-R$ and $x=a+R$ the test often becomes inconclusive; substitute each endpoint into the original series. The two endpoint behaviours may differ.

8. Choose and justify a method

Choose and justify a method

Within the open interval of convergence, termwise differentiation and integration preserve the radius. They can change whether endpoints are included. Start from the geometric series $1/(1-x) = \sum x^n$ for $|x| < 1$, then integrate from zero to x to obtain $-\ln(1-x) = \sum x^n/n$ for $n \geq 1$. Check the integration constant and the real logarithm domain.

9. Choose and justify a method

Choose and justify a method

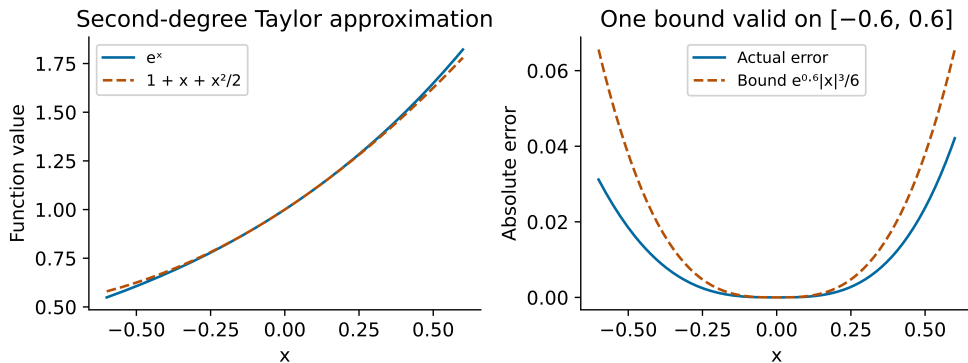
Series also resolve removable limit forms. To evaluate $(e^x - 1 - x)/x^2$ near zero, retain the first surviving term $x^2/2$ rather than using only $e^x \approx 1 + x$. An asymptotic truncation establishes the limit; a finite-interval inequality needs a separate remainder sign or magnitude argument. Do not substitute into a series outside its convergence interval.

10. Worked reasoning

Worked reasoning

For $\sum x^n/n$ with $n \geq 1$, the ratio test gives radius 1. At $x=1$ it is the divergent harmonic series; at $x=-1$ it is an alternating convergent series. Thus its real convergence interval is $[-1,1)$. Differentiating inside gives $\sum x^{n-1} = 1/(1-x)$, which converges at neither endpoint. The radius stayed 1 while the endpoint inclusion changed.

Figure 11: Worked reasoning



Check the labelled values and conditions.

12. Conditions and counterexamples

Conditions and counterexamples

A radius is not a complete interval of convergence. Endpoint tests and factorial coefficients must be checked explicitly.

13. Try independently

Try independently

Find the convergence interval of $\sum_{n=1}^{\infty} x^n/n$ and of its termwise derivative.
Approximate $e^{0.1}$ by its quadratic Taylor polynomial with a justified error bound.

14. Reasoning and check

Reasoning and check

The ratio test gives radius one. At 1 the original harmonic series diverges; at -1 the alternating series converges. Thus the original interval is $[-1, 1)$. The derivative is $\sum_{n=1}^{\infty} x^{n-1} = 1/(1-x)$ only for $|x| < 1$; its terms do not tend to zero at either endpoint. $P_2(0.1) = 1 + 0.1 + 0.1^2/2 = 1.105$. On $[0, 0.1]$, the third derivative is $e^x < 2$. The Lagrange bound gives $|R_2| \leq 2(0.1)^3/6 = 1/3000$. The bound is deliberately conservative, not a claim of exact error.

15. Try independently

Try independently

Evaluate $\lim_{x \rightarrow 0} (\cos x - 1 + x^2/2)/x^4$. Explain why replacing cosine by only $1 - x^2/2$ cannot determine this limit.

16. Reasoning and check

Reasoning and check

Taylor's theorem near zero gives $\cos x = 1 - x^2/2 + x^4/24 + O(x^6)$. Subtraction cancels the constant and quadratic terms, leaving $x^4/24 + O(x^6)$. After division the limit is $1/24$. A truncation with remainder merely $o(x^2)$ gives no control after division by x^4 ; retain and bound the first surviving order.

Check yourself

Reconstruct the method without looking back

Evaluate $\lim_{x \rightarrow 0} (\cos x - 1 + x^2/2)/x^4$. Explain why replacing cosine by only $1 - x^2/2$ cannot determine this limit.