T.5 · Open sets, compactness and connectedness

Original classroom teaching; not an official form or score predictor

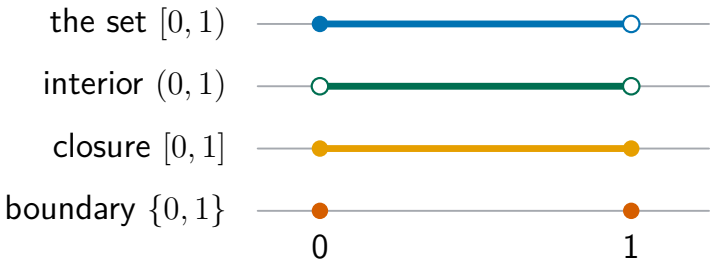
Open sets, compactness and connectedness

- Find the interior, closure and boundary of a set.
- Use the **relative topology 相对拓扑** of a subspace.
- Decide whether a set is compact.
- Decide **connectedness 连通性** and path-connectedness.

Open and closed

- A set is open if every point has a small ball around it inside the set. It is closed if its complement is open, equivalently if it contains all its limit points.
- Open and closed are not opposites: $\mathbb{R}$ and $\varnothing$ are both; $[0, 1)$ is neither; $\mathbb{Q}$ is neither.
- Interior: the largest open subset. Closure: the smallest closed superset. Boundary: closure minus interior.

Figure: Open and closed

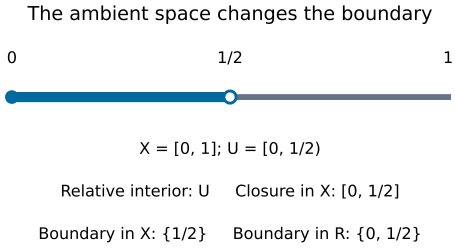


- Interior, closure and boundary of [0, 1).
- Match each part of the diagram to the text.

Subspaces

- $[0, \frac{1}{2})$ is open in $X = [0, 1]$, because it equals $X \cap (-1, \frac{1}{2})$, though it is not open in $\mathbb{R}$.
- Interior, closure and boundary must be computed in the stated space.

Figure: Subspaces



- Interior, closure and boundary depend on the space: U's boundary is $\{1/2\}$ in X but $\{0, 1/2\}$ in $\mathbb{R}$
- Match each part of the diagram to the text.

Compactness

- Compact: every open cover has a finite subcover. In a metric space, equivalently every sequence has a subsequence converging in the set.
- In $\mathbb{R}^n$: compact exactly when closed and bounded. The Cantor set is compact.
- A closed subset of a compact set is compact; a compact subset of a metric space is closed.
- Continuous images of compact sets are compact.

Connectedness

- Connected: the set cannot be split into two disjoint nonempty relatively open parts. In $\mathbb{R}$ the connected sets are the intervals.
- Path-connected (any two points joined by a continuous path) implies connected. The converse fails: the topologist's sine curve, the graph of $\sin(1/x)$ for $x > 0$ together with the segment $\{0\} \times [-1, 1]$, is connected but not path-connected.
- Continuous images of connected sets are connected. The Cantor set is totally disconnected: its only connected pieces are single points.

Worked example: a relative set

- $A = X \cap (-1, 1)$, so A is open in X, and its interior in X is A itself. The point 0 is interior because a small ball in X around 0 has no negative part.
- Its closure is $[0, 1]$ and its boundary in X is $\{1\}$. A is not closed in X: $1 - 1/n \to 1 \notin A$.
- In $\mathbb{R}$ the interior and boundary would be $(0, 1)$ and $\{0, 1\}$.

Worked example: the Cantor set

- The Cantor set is the set of numbers in $[0, 1]$ with a base-3 expansion using only the digits 0 and 2.
- $\frac{1}{4} = 0.020202\ldots_3$ is in the set, though it is not an endpoint of any removed interval.
- It is closed (an intersection of closed sets), bounded, so compact; uncountable; of length zero; and contains no interval.

Common errors

- Treating "not open" as "closed".
- Computing interior or boundary in $\mathbb{R}$ when the question names a subspace.
- Using "closed and bounded" for compactness outside $\mathbb{R}^n$.

Common errors (2)

- Assuming connected sets are path-connected.
- Thinking a set of length zero must be countable.

Key terms

- **relative topology**: The open sets of a subspace, formed by intersecting it with open sets of the larger space.
- **connectedness**: The property of a space that cannot be split into two disjoint nonempty open parts.

Check yourself

- Is $\mathbb{Q}$ open in $\mathbb{R}$? Closed?
- Is $[0,1] \cup [2,3]$ connected?
- A continuous function maps $[0,1]$ into $\mathbb{R}$. What kind of set is the image?