T.4 · Sequences, series and uniform convergence

Original classroom teaching; not an official form or score predictor

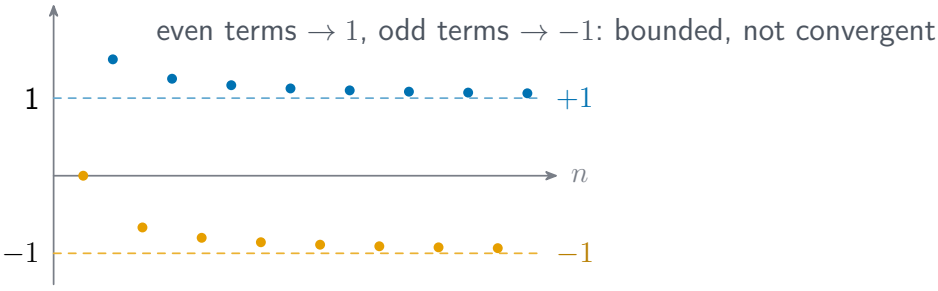
Sequences, series and uniform convergence

- Decide whether a sequence converges, using monotonicity and the **Cauchy sequence 柯西序列** idea.
- Choose and apply a convergence test for a series.
- Separate absolute from conditional convergence.
- Tell pointwise from uniform convergence, and know what uniform convergence preserves.

Sequences

- A convergent sequence is bounded; a bounded sequence need not converge: $(-1)^n + 1/n$ has subsequences tending to 1 and -1.
- A bounded monotone sequence converges.
- In $\mathbb{R}$, a sequence converges exactly when it is a **Cauchy sequence**.

Figure: Sequences



- Bounded, not convergent: two subsequential limits.
- Read the axes and units first.

Tests for series

- nth-term test: if $a_n \not\to 0$, the series diverges. If $a_n \to 0$, nothing follows ($\sum 1/n$ diverges).
- Comparison: compare with $\sum 1/n^p$ (converges for $p > 1$) or a geometric series.
- Ratio test: if $|a_{n+1}/a_n| \to L$, the series converges absolutely for $L < 1$ and diverges for $L > 1$. $L = 1$ decides nothing.
- Integral test for positive decreasing terms; alternating series test for alternating terms that decrease to 0.

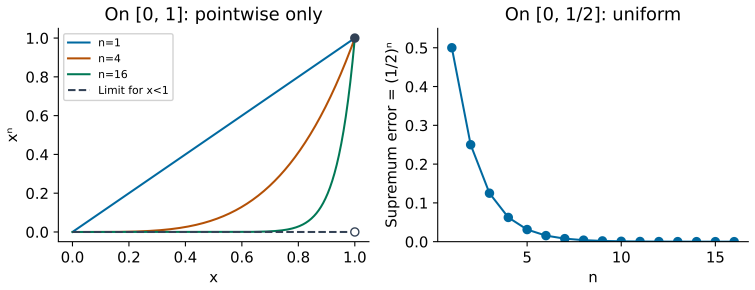
Absolute and conditional

- $\sum a_n$ converges absolutely when $\sum |a_n|$ converges; absolute convergence implies convergence.
- $1 - \frac{1}{2} + \frac{1}{3} - \cdots$ converges (to $\ln 2$) but not absolutely: conditional convergence.
- Riemann: a conditionally convergent series can be rearranged to sum to any real number.

Uniform convergence

- A uniform limit of continuous functions is continuous. So if the limit is discontinuous, the convergence was not uniform.
- Uniform convergence on a bounded interval allows integrating term by term.
- Weierstrass M-test: if $|f_n(x)| \leq M_n$ for all x and $\sum M_n$ converges, then $\sum f_n$ converges uniformly.

Figure: Uniform convergence



- x^n converges only pointwise on [0, 1], but uniformly on [0, 1/2]
- Match each part of the diagram to the text.

Worked example: powers of x

- On $[0,1]$ the limit is 0 for $x<1$ and 1 at $x=1$. The limit is discontinuous, so the convergence is not uniform; indeed $\sup_{[0,1)} x^n = 1$.
- On $[0,a]$, $\sup x^n = a^n \to 0$: uniform convergence to 0.
- The domain, not just the formula, decides uniformity.

Worked example: choosing tests

- $\sum \dfrac{n!}{n^n}$: the ratio is $\left(\dfrac{n}{n+1}\right)^n \to \dfrac{1}{e} < 1$, so it converges.
- $\sum \dfrac{\sin nx}{n^2}$: $|\sin nx/n^2| \le 1/n^2$, so by the M-test it converges uniformly on $\mathbb{R}$, and its sum is continuous.
- $\sum \dfrac{(-1)^n}{\sqrt{n}}$: converges by the alternating test, but $\sum 1/\sqrt{n}$ diverges, so the convergence is conditional.

Common errors

- Concluding convergence from $a_n \to 0$.
- Treating ratio limit 1 as a decision.
- Assuming that pointwise convergence preserves continuity or integrals.

Common errors (2)

- Rearranging a conditionally convergent series and keeping the sum.
- Forgetting that uniform convergence depends on the domain.

Key terms

- **Cauchy sequence**: A sequence whose terms eventually stay arbitrarily close to one another.
- **uniform convergence**: Convergence in which one index makes the error small at every point of the domain at once.

Check yourself

- Does $\sum \dfrac{n}{2^n}$ converge, and to what?
- Does $f_n(x) = x/n$ converge uniformly on $\mathbb{R}$?
- Does $\sum \dfrac{1}{n \ln n}$ (from $n = 2$) converge?