

T.4 · Sequences, series and uniform convergence

Original classroom teaching; not an official form or score predictor

1. Scope and prerequisites

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

2. Scope and prerequisites

Scope and prerequisites

Prerequisites: Sequence limits, supremum, continuity and numerical series.

3. Scope and prerequisites

Scope and prerequisites

Use Cauchy, monotone convergence and subsequence criteria Distinguish pointwise from uniform convergence of functions Check hypotheses before interchanging limits and integrals

4. Scope and prerequisites

Scope and prerequisites

Cauchy sequence : A sequence whose sufficiently late terms are arbitrarily close to one another.

5. Scope and prerequisites

Scope and prerequisites

uniform convergence : Convergence with one error threshold index valid at every point of the domain.

6. Choose and justify a method

Choose and justify a method

A convergent real sequence is Cauchy, and every real Cauchy sequence converges because $\mathbb{R}$ is complete. In $\mathbb{Q}$ a Cauchy sequence can approach an irrational number and fail to converge within $\mathbb{Q}$. A bounded monotone real sequence converges. A bounded sequence need not converge, but Bolzano–Weierstrass guarantees a convergent subsequence; $(-1)^n$ has two different subsequential limits.

7. Choose and justify a method

Choose and justify a method

For a numerical series, absolute convergence implies convergence; conditional convergence does not allow arbitrary rearrangement without affecting the sum. The alternating harmonic series converges but its absolute-value series diverges. In a ratio test, a limit below 1 proves absolute convergence and one above 1 proves divergence; a limit equal to 1 is inconclusive, as both $\sum 1/n$ and $\sum 1/n^2$ demonstrate.

8. Choose and justify a method

Choose and justify a method

Pointwise convergence chooses an index N separately for each x and tolerance. Uniform convergence chooses one N that works for every x in the domain. For real-valued functions, check the supremum of $|f_n - f|$ over the whole domain. A continuous pointwise limit does not by itself prove uniform convergence. Domain endpoints and shrinking peaks often distinguish the two notions.

9. Choose and justify a method

Choose and justify a method

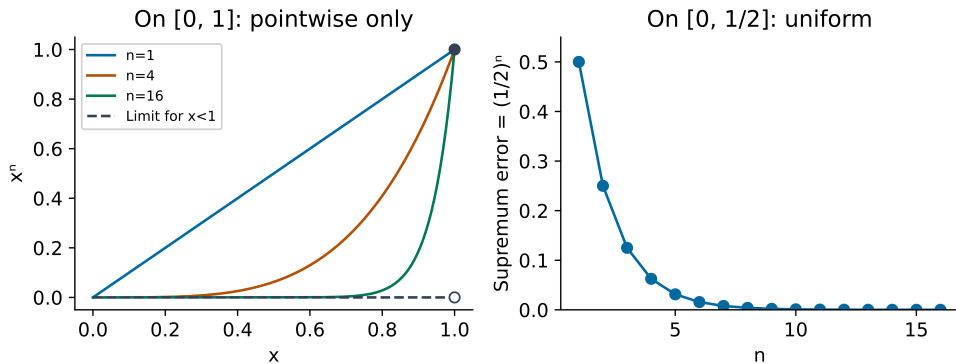
A uniform limit of continuous functions is continuous. On a closed bounded interval, uniform convergence of Riemann-integrable functions permits exchanging limit and integral. Exchanging derivatives needs extra hypotheses; uniform convergence of the functions alone is insufficient. The Weierstrass M-test establishes uniform absolute convergence of a series of functions if each term is bounded by M_n on the whole domain and $\sum M_n$ converges.

10. Worked reasoning

Worked reasoning

On $[0,1]$, $f_n(x)=x^n$ tends to 0 for $x<1$ and to 1 at $x=1$. The limit is discontinuous, so convergence cannot be uniform; directly, the supremum error is 1, approached below 1. On $[0,a]$ with $0\leq a<1$, the limit is zero and the supremum is a^n , which tends to zero, so convergence is uniform. Changing the domain changes the conclusion.

Figure 11: Worked reasoning



Check the labelled values and conditions.

12. Conditions and counterexamples

Conditions and counterexamples

Checking several fixed x values proves neither a supremum bound nor uniform convergence. The point producing the largest error may move with n .

13. Try independently

Try independently

Determine pointwise and uniform convergence of $f_n(x) = x^n$ on $[0, 1]$ and on $[0, a]$ for fixed $0 < a < 1$. Compare $\sum 1/n^2$ with the alternating harmonic series for absolute convergence.

14. Reasoning and check

Reasoning and check

On $[0, 1]$, the limit is zero for $x < 1$ and one at 1. This limit is discontinuous, so continuous f_n cannot converge uniformly. More directly the supremum error on $[0, 1)$ is one. On $[0, a]$ the supremum is $a^n \rightarrow 0$, so convergence is uniform to zero. The p-series with $p=2$ converges absolutely. The alternating harmonic series converges by decreasing terms tending to zero, but the absolute series diverges; its convergence is conditional.

15. Try independently

Try independently

Let $g_n(x) = \sin(nx)/n$ on $\mathbb{R}$. Is convergence uniform? May its derivatives converge to the derivative of its limit? Use an explicit point to decide.

16. Reasoning and check

Reasoning and check

$\sup_{x \in \mathbb{R}} |g_n(x)| = 1/n \rightarrow 0$, so the functions converge uniformly to zero. But $g'_n(x) = \cos(nx)$, and at $x = 0$ these derivatives are all 1. The derivative of the zero limit is 0. Thus uniform convergence of the functions alone does not justify differentiating a limit; additional derivative-convergence hypotheses are needed.

Check yourself

Reconstruct the method without looking back

Let $g_n(x) = \sin(nx)/n$ on $\mathbb{R}$. Is convergence uniform? May its derivatives converge to the derivative of its limit? Use an explicit point to decide.