A.4 · Rings, ideals and modules

Original classroom teaching; not an official form or score predictor

Rings, ideals and modules

- Tell rings, integral domains and fields apart.
- Find units and zero divisors.
- Work with ideals and quotient rings.

Rings, ideals and modules (2)

- Decide whether an ideal is prime or maximal.
- Recognise **torsion** 挠性 in modules.

Rings, domains and fields

- A ring has addition and multiplication; here rings are commutative with 1.
- An integral domain has no zero divisors: $ab = 0$ forces $a = 0$ or $b = 0$. $\mathbb{Z}$ and $\mathbb{R}[x]$ are domains.
- A field is a ring in which every nonzero element is a unit. Every field is a domain; every finite domain is a field.
- In $\mathbb{Z}/n\mathbb{Z}$, the units are the residues coprime to n, and the other nonzero residues are zero divisors.

Ideals and quotients

- The principal ideal (a) is all multiples of a. Every ideal of $\mathbb{Z}$ is (n).
- The quotient ring R/I has the cosets $r + I$, with the operations of R.
- $\mathbb{Z}/(n) = \mathbb{Z}/n\mathbb{Z}$; $\mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$.

Prime and maximal ideals

- P is prime when R/P is a domain: $ab \in P$ forces $a \in P$ or $b \in P$.
- M is maximal when R/M is a field: no ideal lies strictly between M and R.
- Every maximal ideal is prime. In $\mathbb{Z}$, (p) for a prime p is maximal; (0) is prime but not maximal.
- Unique factorisation fails in $\mathbb{Z}[\sqrt{-5}]$: $6 = 2 \times 3 = (1+\sqrt{-5})(1-\sqrt{-5})$.

Figure: Prime and maximal ideals

Multiplication in Z/6Z: units and zero divisors

Residue	Product mod 6	Classification
1, 5	1·1 = 1; 5·5 = 1	Units
2, 3, 4	2·3 = 0; 4·3 = 0	Nonzero zero divisors
0	0·a = 0	Zero

- In $\mathbb{Z}/6\mathbb{Z}$ a residue is a unit exactly when it is coprime to 6
- Match each part of the diagram to the text.

Modules

- A module is like a vector space, but its scalars come from a ring.
- An element m is **torsion** if some nonzero scalar r gives $rm = 0$. In the $\mathbb{Z}$-module $\mathbb{Z}/6\mathbb{Z}$, every element is torsion, since $6m = 0$.
- $\mathbb{Z}$ as a $\mathbb{Z}$-module has no nonzero torsion elements.

Worked example: Z/12

- Units: residues coprime to 12, namely 1, 5, 7, 11.
- Zero divisors: 2, 3, 4, 6, 8, 9, 10. For example $4 \times 3 = 0$.
- The ideal $(4) = \{0, 4, 8\}$ has quotient $\cong \mathbb{Z}/4\mathbb{Z}$, which has $2 \times 2 = 0$. So (4) is neither prime nor maximal.

Worked example: polynomial quotients

- x^2+1 has no real root and degree 2, so it is irreducible over $\mathbb{R}$. $\mathbb{R}[x]/(x^2+1)$ is a field (it is $\mathbb{C}$), and the ideal is maximal.
- Over $\mathbb{C}$, $x^2+1=(x-i)(x+i)$, so $\mathbb{C}[x]/(x^2+1)$ has zero divisors; the ideal is not even prime.
- In $\mathbb{Z}[x]$, the ideal (x) has quotient $\mathbb{Z}$, a domain but not a field: prime, not maximal.

Common errors

- Calling every domain a field.
- Forgetting that a maximal ideal is defined relative to the whole ring.
- Assuming unique factorisation holds in every domain.

Common errors (2)

- Treating "prime ideal" and "ideal generated by a prime number" as always the same.
- Missing **torsion** in a finite module.

Key terms

- **ideals**: Additive subgroups of a ring that absorb multiplication by every ring element.
- **torsion**: The property of a module element that some nonzero scalar sends it to zero.

Check yourself

- Is (6) a prime ideal of $\mathbb{Z}$?
- How many units does $\mathbb{Z}/15\mathbb{Z}$ have?
- Is $\mathbb{Q}[x]/(x^2-2)$ a field?