

A.4 · Rings, ideals and modules

Original classroom teaching; not an official form or score predictor

1. Scope and prerequisites

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

2. Scope and prerequisites

Scope and prerequisites

Prerequisites: Rings, fields, additive subgroups and homomorphisms.

3. Scope and prerequisites

Scope and prerequisites

Distinguish units, zero divisors and integral domains Identify ideals and interpret quotient rings Compare modules over rings with vector spaces over fields Derive Boolean-ring properties without assuming commutativity

4. Scope and prerequisites

Scope and prerequisites

ideal : An additive subgroup of a ring absorbing multiplication by all ring elements.

5. Scope and prerequisites

Scope and prerequisites

torsion : An element is annihilated by a nonzero scalar in the stated module.

6. Choose and justify a method

Choose and justify a method

For the domain, ideal and quotient examples below, use a commutative ring with identity 1 distinct from 0. A source problem may specify a general ring instead; do not assume its multiplication commutes unless stated or proved. A unit has a multiplicative inverse. A nonzero zero divisor multiplies some nonzero element to zero. An integral domain has no such zero divisors; cancellation of a nonzero factor then works. A field is a domain in which every nonzero element is a unit. $\mathbb{Z}$ is a domain but not a field; $\mathbb{Z}/6\mathbb{Z}$ is neither.

7. Choose and justify a method

Choose and justify a method

An ideal I is an additive subgroup that absorbs multiplication by every ring element. This is stronger than being a subring. In $\mathbb{Z}$, $n\mathbb{Z}$ is an ideal; quotient elements are integer residue classes modulo n . In a commutative ring, R/I is a field exactly when I is maximal, and it is a domain exactly when I is prime. The ideal must be proper in both statements.

8. Choose and justify a method

Choose and justify a method

A module allows scalars from a ring instead of requiring a field. Every abelian group is a $\mathbb{Z}$ -module by repeated addition, but it need not have a vector-space basis. In $\mathbb{Z}/6\mathbb{Z}$ as a $\mathbb{Z}$ -module, 6 times the nonzero residue 1 is zero; this is torsion. For a vector space over a field, a nonzero scalar is invertible and cannot annihilate a nonzero vector.

9. Choose and justify a method

Choose and justify a method

A submodule is closed under addition and all permitted scalar actions. A linear map of modules preserves both. The kernel and image are submodules, and the quotient by the kernel is isomorphic to the image. Do not apply finite-dimensional rank-nullity to an arbitrary module without establishing an appropriate free-module setting; integer row operations and field row operations permit different divisions. In a general Boolean ring, every a satisfies $a^2=a$. Do not assume commutativity to prove it: idempotence of $a+a$ gives $4a=2a$, hence $2a=0$.

10. Choose and justify a method

Choose and justify a method

Expanding $(a+b)^2=a+b$ gives $ab+ba=0$, and characteristic two makes $-ba=ba$; therefore $ab=ba$. Idempotence does not imply nilpotence: in F_2 , the nonzero element 1 satisfies $1^n=1$ for every positive n .

11. Worked reasoning

Worked reasoning

In $\mathbb{Z}/6\mathbb{Z}$, $2 \cdot 3 = 0$, so 2 and 3 are zero divisors. The units are 1 and 5 because their gcd with 6 is 1. In $\mathbb{Z}$, the ideal $5\mathbb{Z}$ gives the field $\mathbb{Z}/5\mathbb{Z}$, while $6\mathbb{Z}$ gives a quotient with zero divisors. As $\mathbb{Z}$ -modules, the map from $\mathbb{Z}$ to $\mathbb{Z}/6\mathbb{Z}$ has kernel $6\mathbb{Z}$; the quotient identifies integers differing by a multiple of 6, rather than producing a real vector space.

Figure 12: Worked reasoning

Multiplication in $\mathbb{Z}/6\mathbb{Z}$: units and zero divisors

Residue	Product mod 6	Classification
1, 5	$1 \cdot 1 = 1$; $5 \cdot 5 = 1$	Units
2, 3, 4	$2 \cdot 3 = 0$; $4 \cdot 3 = 0$	Nonzero zero divisors
0	$0 \cdot a = 0$	Zero

Check the labelled values and conditions.

13. Conditions and counterexamples

Conditions and counterexamples

A set can be a submodule without being a vector space over the rationals. Scalar division is valid only when the scalar inverse belongs to the structure.

14. Try independently

Try independently

In $\mathbb{Z}/12\mathbb{Z}$, list the units and nonzero zero divisors. Is the ideal (4) prime or maximal?

15. Reasoning and check

Reasoning and check

Units are the residues coprime to 12: 1, 5, 7, 11 . The remaining nonzero residues 2, 3, 4, 6, 8, 9, 10 are zero divisors. For each such a , $a \cdot (12/\gcd(a, 12)) = 0$ modulo 12, with the second factor nonzero. The ideal $(4) = \{0, 4, 8\}$ gives quotient isomorphic to $\mathbb{Z}/4\mathbb{Z}$. This is not a domain, since $2 \cdot 2 = 0$, and is not a field. Hence the ideal is neither prime nor maximal.

16. Try independently

Try independently

As a $\mathbb{Z}$ -module, can $\mathbb{Z}/6\mathbb{Z}$ be free? Compare the ideals (2) and (x) in $\mathbb{Z}$ and $\mathbb{Q}[x]$, respectively, using their quotients.

17. Reasoning and check

Reasoning and check

A nonzero free $\mathbb{Z}$ -module has no nonzero vector annihilated by a nonzero integer: examine each integer coordinate. But $6[1] = 0$ in $\mathbb{Z}/6\mathbb{Z}$. It is not free; a field-style basis argument is invalid. $\mathbb{Z}/(2) \cong \mathbb{F}_2$ is a field, so (2) is maximal and prime. Evaluation at zero maps $\mathbb{Q}[x]$ onto $\mathbb{Q}$ with kernel (x) ; its quotient is a field too. Both arguments use proper ideals in commutative rings with identity.

Check yourself

Reconstruct the method without looking back

As a $\mathbb{Z}$ -module, can $\mathbb{Z}/6\mathbb{Z}$ be free? Compare the ideals (2) and (x) in $\mathbb{Z}$ and $\mathbb{Q}[x]$, respectively, using their quotients.