A.3 · Groups, cosets and quotient maps

Original classroom teaching; not an official form or score predictor

Groups, cosets and quotient maps

- List subgroups and cosets, and use Lagrange's theorem and the index.
- Decide whether a subgroup is a **normal subgroup** 正规子群.
- Build quotient groups and identify them.
- Use the first isomorphism theorem.

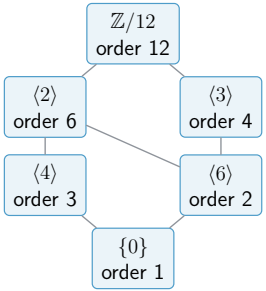
Cosets

- The left cosets partition G into pieces of equal size $|H|$.
- So $|G| = [G : H]\,|H|$, where the index $[G : H]$ is the number of cosets. This is Lagrange's theorem.
- In additive notation the coset is $g + H$.

Cyclic groups

- Every subgroup of a cyclic group is cyclic.
- $\mathbb{Z}/n\mathbb{Z}$ has exactly one subgroup of each order d dividing n, namely $\langle n/d \rangle$.
- $\mathbb{Z}/12\mathbb{Z}$ has subgroups of orders 1, 2, 3, 4, 6 and 12.

Figure: Cyclic groups



cyclic: exactly one subgroup per divisor; in a general group Lagrange's converse can fail

- Subgroups of a cyclic group of order 12.
- Match each part of the diagram to the text.

Normal subgroups and quotients

- Every subgroup of an abelian group is normal. A subgroup of index 2 is always normal.
- If H is normal, the cosets form a group G/H, with $(aH)(bH) = abH$ and order $[G : H]$.
- If H is not normal, this product is not well defined.

The first isomorphism theorem

- To identify a quotient G/H, find an onto homomorphism with kernel H.
- Reduction mod 3 maps $\mathbb{Z}/12\mathbb{Z}$ onto $\mathbb{Z}/3\mathbb{Z}$ with kernel $\langle 3 \rangle$, so $(\mathbb{Z}/12\mathbb{Z})/\langle 3 \rangle \cong \mathbb{Z}/3\mathbb{Z}$.

$$G/\ker\phi \cong \operatorname{im}\phi$$

Figure: The first isomorphism theorem

Reduction modulo 3: three fibres, four elements each

Quotient class	Elements of Z/12Z	Image in Z/3Z
H	0, 3, 6, 9	0
1 + H	1, 4, 7, 10	1
2 + H	2, 5, 8, 11	2

- Reduction mod 3 splits $\mathbb{Z}/12\mathbb{Z}$ into three fibres of four elements each
- Match each part of the diagram to the text.

Worked example: a quotient of Z/18

- $H = \{0, 6, 12\}$, so 6 has order 3.
- The cosets are $r + H = \{r, r+6, r+12\}$ for $r = 0, 1, \ldots, 5$: six cosets of three elements, covering all 18 residues.
- Reduction mod 6 is onto $\mathbb{Z}/6\mathbb{Z}$ with kernel H, so $(\mathbb{Z}/18\mathbb{Z})/H \cong \mathbb{Z}/6\mathbb{Z}$.

Worked example: a subgroup that is not normal

- $(13)H = \{(13), (13)(12)\} = \{(13), (123)\}$.
- $H(13) = \{(13), (12)(13)\} = \{(13), (132)\}$.
- The left and right cosets differ, so H is not normal. By contrast $A_3 = \{e, (123), (132)\}$ has index 2 and is normal.

Common errors

- Assuming every subgroup is normal.
- Forming a quotient by a subgroup that is not normal.
- Confusing the order of an element with the order of the quotient.

Common errors (2)

- Composing permutations in the wrong order.
- Thinking a group must have a subgroup for every divisor of its order (only cyclic groups promise this).

Key terms

- **cosets**: The translates gH of a subgroup, which split a group into equal parts.
- **normal subgroup**: A subgroup unchanged by conjugation, $gHg^{-1} = H$, for every group element.

Check yourself

- How many cosets does $\langle 4 \rangle$ have in $\mathbb{Z}/20\mathbb{Z}$?
- A subgroup has index 2. Must it be normal?
- What is $(\mathbb{Z}/12\mathbb{Z})/\langle 4 \rangle$ isomorphic to?