

A.3 · Groups, cosets and quotient maps

Original classroom teaching; not an official form or score predictor

1. Scope and prerequisites

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

2. Scope and prerequisites

Scope and prerequisites

Prerequisites: Group axioms, functions, integer divisibility and permutations.

3. Scope and prerequisites

Scope and prerequisites

Verify group structure and compute element orders and subgroup indices Use kernels and images to identify quotient groups Distinguish normal subgroups from arbitrary subgroups Classify permutation conjugacy by cycle type

4. Scope and prerequisites

Scope and prerequisites

coset : A translate of a subgroup that forms one part of the coset partition.

5. Scope and prerequisites

Scope and prerequisites

normal subgroup element. : A subgroup invariant under conjugation by every group element.

6. Choose and justify a method

Choose and justify a method

Before computing an order, check closure, associativity, an identity and an inverse for every element under the stated operation. A subset can inherit associativity yet fail closure or omit the identity. In a finite group, the order of an element is the least positive power giving the identity. In the additive group $\mathbb{Z}/n\mathbb{Z}$, it is the least positive multiple giving zero; the order of residue a is $n/\gcd(a,n)$. Lagrange's theorem says subgroup orders divide the group order.

7. Choose and justify a method

Choose and justify a method

The converse is not a general existence theorem, and the order of a group is not the order of each element.

8. Choose and justify a method

Choose and justify a method

A left coset gH is a translate of a subgroup H . Cosets have equal size and partition the group, so the index is $|G|/|H|$ in a finite group. In an additive group write $g+H$.

Membership in the same coset means the difference lies in H . A coset usually is not itself a subgroup because it may omit the identity.

9. Choose and justify a method

Choose and justify a method

A homomorphism preserves the operation. Its kernel consists of elements sent to the identity, and its image consists of values actually reached. Every kernel is normal. The first isomorphism theorem identifies $G/\ker(\phi)$ with $\text{im}(\phi)$; do not replace the image with the whole codomain unless the map is onto.

10. Choose and justify a method

Choose and justify a method

Quotient multiplication is well defined only when H is normal. All subgroups of an abelian group are normal. In a nonabelian group test $gHg^{-1} = H$; a subgroup of index two is normal. For permutations compose in the stated convention, here rightmost first. Disjoint cycle lengths give the permutation order by their least common multiple. Conjugation $h\sigma h^{-1}$ relabels the elements in σ 's cycles, so it preserves cycle lengths; conversely permutations with the same cycle lengths can be related by a relabelling.

11. Choose and justify a method

Choose and justify a method

Thus conjugacy classes in S_n correspond to partitions of n , including fixed-point cycles. In S_4 the types are $1+1+1+1$, $2+1+1$, $2+2$, $3+1$ and 4 : five classes, not one class for each possible element order. The types $2+1+1$ and $2+2$ both have order two but are not conjugate.

12. Worked reasoning

Worked reasoning

Define ϕ from $\mathbb{Z}/12\mathbb{Z}$ to $\mathbb{Z}/3\mathbb{Z}$ by reducing residues modulo 3. It is onto and preserves addition. Its kernel H is $\{0, 3, 6, 9\}$, so $|H|=4$ and the index is $12/4=3$. The other cosets are $\{1, 4, 7, 10\}$ and $\{2, 5, 8, 11\}$. Thus $(\mathbb{Z}/12\mathbb{Z})/H$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$. The element 3 in the original group has order $12/\gcd(3, 12)=4$, not 3.

Figure 13: Worked reasoning

Reduction modulo 3: three fibres, four elements each

Quotient class	Elements of $\mathbb{Z}/12\mathbb{Z}$	Image in $\mathbb{Z}/3\mathbb{Z}$
H	0, 3, 6, 9	0
$1 + H$	1, 4, 7, 10	1
$2 + H$	2, 5, 8, 11	2

Check the labelled values and conditions.

14. Conditions and counterexamples

Conditions and counterexamples

A quotient has one element per coset, not one per element of its kernel. A homomorphism need not be onto its stated codomain.

15. Try independently

Try independently

In the additive group $\mathbb{Z}/18\mathbb{Z}$, let $H = \langle 6 \rangle$. List H and its cosets. Determine the order of 6 and identify the quotient.

16. Reasoning and check

Reasoning and check

$H = \{0, 6, 12\}$ has size three. The six cosets are $r + H = \{r, r + 6, r + 12\}$ for $r = 0, 1, 2, 3, 4, 5$. They partition all 18 residues. The element 6 has order three, while the quotient has order six. Reduction modulo six is onto with kernel H , so $(\mathbb{Z}/18\mathbb{Z})/H \cong \mathbb{Z}/6\mathbb{Z}$. Normality holds because the original group is abelian.

17. Try independently

Try independently

In S_3 , let $H = \{e, (12)\}$. Check whether H is normal, and decide whether coset multiplication defines a quotient group. Separately find the order of $(123)(45)$ in S_5 .

18. Reasoning and check

Reasoning and check

Conjugation by (123) sends (12) to (23) , which is outside H . Thus H is not normal. Left cosets still partition S_3 , but multiplication of those cosets is not well defined independently of representatives. The disjoint cycles in $(123)(45)$ have lengths three and two, so its order is $\text{lcm}(3, 2) = 6$. Element order and subgroup index are different quantities.

Check yourself

Reconstruct the method without looking back

In S_3 , let $H = \{e, (12)\}$. Check whether H is normal, and decide whether coset multiplication defines a quotient group. Separately find the order of $(123)(45)$ in S_5 .