C.3 · Differential equations and initial conditions

Original classroom teaching; not an official form or score predictor

Differential equations and initial conditions

- Solve separable and first-order linear equations.
- Solve second-order linear equations with constant coefficients from the **characteristic root 特征根**.
- Use an **initial condition 初始条件** to fix constants.
- Know when a solution exists and is unique.

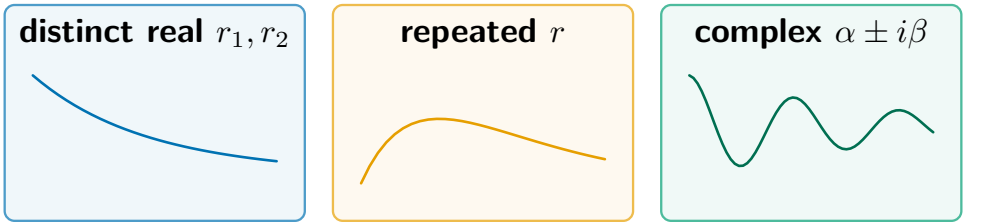
First-order equations

- Separable: $\dfrac{dy}{dx} = g(x)h(y)$. Separate and integrate: $\displaystyle\int \frac{dy}{h(y)} = \int g(x)\,dx$.
- Linear: $y' + p(x)y = q(x)$. Multiply by the integrating factor $\mu = e^{\int p\,dx}$; then $(\mu y)' = \mu q$.
- Growth and decay: $y' = ky$ gives $y = y_0 e^{kt}$.

Second-order with constant coefficients

- Two distinct real roots: $y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$.
- A repeated root r: $y = (C_1 + C_2 t) e^{rt}$.
- Complex roots $\alpha \pm i\beta$: $y = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$.
- A second-order equation needs two conditions, such as $y(0)$ and $y'(0)$.

Figure: Second-order with constant coefficients



- The roots decide the solution form.
- Match each part of the diagram to the text.

Existence and uniqueness

- If f and $\partial f/\partial y$ are continuous near (x_0, y_0), there is exactly one solution near x_0.
- Without this, uniqueness can fail: $y' = 3y^{2/3}$, $y(0) = 0$ has both $y = 0$ and $y = x^3$.
- Solutions can also blow up: $y' = y^2$, $y(0) = 1$ gives $y = 1/(1 - x)$, which exists only for $x < 1$.

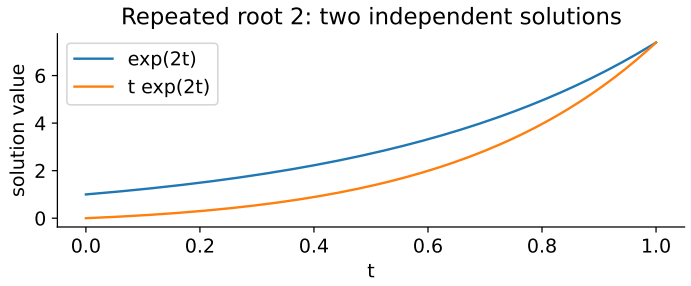
Checking a solution

- Substitute it back into the equation, and check every initial condition.
- Count constants: an nth-order equation's general solution has n of them.

Worked example: a repeated root

- $r^2 - 4r + 4 = (r-2)^2$, a repeated root 2, so $y = (C_1 + C_2 t)e^{2t}$.
- $y(0) = C_1 = 1$; $y'(0) = 2C_1 + C_2 = 0$, so $C_2 = -2$.
- $y = (1 - 2t)e^{2t}$. Check: $y' = -4te^{2t}$, $y'' = (-4 - 8t)e^{2t}$, and $y'' - 4y' + 4y = 0$.

Figure: Worked example: a repeated root



- A repeated root 2 gives the two independent solutions $e^{(2t)}$ and $t \cdot e^{(2t)}$
- Read the axes and units first.

Worked example: cooling

- $T - 20 = 70e^{-kt}$. At $t = 10$: $40 = 70e^{-10k}$, so $k = \ln(7/4)/10 = 0.0560$ per minute.
- For 30 °C: $10 = 70e^{-kt}$, so $t = \ln 7/k = 1.946/0.0560 = 34.8$ minutes.
- The tea never reaches 20 °C exactly; the difference only decays.

Common errors

- Using two copies of e^{rt} for a repeated root; the second solution is te^{rt}.
- Forgetting the integrating factor's effect on the right-hand side.
- Fixing only one constant of a second-order equation.

Common errors (2)

- Assuming every initial value problem has a unique solution.
- Dividing by $h(y)$ and losing the constant solution where $h(y) = 0$.

Key terms

- **characteristic root**: A root of the polynomial obtained by substituting e^{rt} into a linear constant-coefficient equation.
- **initial condition**: A value of the solution or its derivative given at a starting point.

Check yourself

- What is the general solution of $y'' + 9y = 0$?
- Solve $y' = 2xy$ with $y(0) = 3$.
- How many constants does the general solution of a third-order linear equation have?