T.3 · Complex analysis and residues

Original classroom teaching; not an official form or score predictor

Complex analysis and residues

- Use the polar form, de Moivre's theorem and roots of unity.
- Test whether a function is **analytic** 解析的 with the Cauchy–Riemann equations.
- Apply Cauchy's theorem and integral formula.
- Find a **residue** 留数 and use the residue theorem.

Polar form

- $z = re^{i\theta}$ with $r = |z|$. Multiplying multiplies moduli and adds arguments.
- De Moivre: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.
- The nth roots of unity are $e^{2\pi ik/n}$, $k = 0, \ldots, n-1$. They sum to 0 for $n \geq 2$.

Analytic functions

- With continuous partial derivatives, these Cauchy–Riemann equations are also sufficient.
- z^2 passes everywhere; $\bar{z}$ fails everywhere, though it is smooth as a map of the plane.
- An analytic function's real and imaginary parts are harmonic: $u_{xx} + u_{yy} = 0$.

$$u_x = v_y, \qquad u_y = -v_x$$

Figure: Analytic functions

need $u_x = v_y$ **and** $u_y = -v_x$

$f = z^2$: $u = x^2 - y^2$,
$v = 2xy$
$u_x = 2x = v_y$
$u_y = -2y = -v_x$
analytic everywhere

$f = \bar{z}$: $u = x$, $v = -y$
$u_x = 1$, $v_y = -1$
real partials exist, but **nowhere complex differentiable**

- Cauchy-Riemann: z squared passes, the conjugate fails.
- Match each part of the diagram to the text.

Cauchy's theorems

- Cauchy's theorem: if f is analytic inside and on a closed curve C, then $\oint_C f\,dz = 0$.
- Integral formula: $f(a) = \dfrac{1}{2\pi i} \oint_C \dfrac{f(z)}{z-a}\,dz$ for a inside C.
- Liouville: a bounded function that is analytic on the whole plane is constant.

Residues

- Simple pole: $\mathrm{Res} = \lim_{z\to a}(z-a)f(z)$.
- Pole of order m: $\mathrm{Res} = \dfrac{1}{(m-1)!}\lim_{z\to a}\dfrac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$.

$$\oint_C f\,dz = 2\pi i \sum \mathrm{Res}(f, a_k) \qquad \text{(anticlockwise, singularities inside } C\text{)}$$

Figure: Residues

The residue is the coefficient of $(z-1)^{-1}$

Function	Pole order	Residue at 1
$3/(z-1)$	1	3
$e^z/(z-1)^2$	2	e
$e^{2z}/(z-1)^3$	3	$2e^2$

- The residue is the coefficient of $(z-1)^{-1}$ in the Laurent series
- Match each part of the diagram to the text.

Worked example: a double pole

- $e^z/z^2 = z^{-2} + z^{-1} + \frac{1}{2} + \cdots$, so the residue at 0 is 1.
- The integral is $2\pi i \times 1 = 2\pi i$.
- A double pole does not mean a zero residue; read the coefficient of z^{-1}.

Worked example: a real integral

- Close the real line with a large semicircle in the upper half-plane; its contribution goes to zero.
- Inside is the simple pole $z = i$, with residue $\left.\dfrac{1}{z+i}\right|_{z=i} = \dfrac{1}{2i}$.
- The integral is $2\pi i \times \dfrac{1}{2i} = \pi$, which agrees with $[\arctan x]_{-\infty}^{\infty}$.

Common errors

- Treating a smooth map of the plane as complex differentiable.
- Reading the residue from the wrong power of $(z-a)$.
- Including poles outside the contour.

Common errors (2)

- Forgetting the factor $\dfrac{1}{(m-1)!}$ for higher-order poles.
- Going around the contour clockwise without changing the sign.

Key terms

- **analytic**: Complex differentiable at every point of an open set.
- **residue**: The coefficient of $(z-a)^{-1}$ in the Laurent series of a function at a.

Check yourself

- What is the sum of the three cube roots of unity?
- What is the residue of $\dfrac{1}{z(z-2)}$ at $z=2$?
- Can a non-constant function that is analytic on the whole plane be bounded?