T.2 · Discrete mathematics, probability and numerical methods

Original classroom teaching; not an official form or score predictor

Discrete mathematics, probability and numerical methods

- Count with permutations, combinations and the complement.
- Set up and solve a **recurrence** 递推关系.
- Use independence, conditional probability and **mutually exclusive** 互斥的 events correctly.

Discrete mathematics, probability and numerical methods (2)

- Use expectation and the binomial distribution.
- Apply simple numerical methods and know their errors.

Counting

- Ordered selections of k from n distinct objects: $n!/(n-k)!$. Unordered: $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$.
- With repetition allowed, strings of length k from n symbols: n^k.
- "At least one" is usually easiest as 1 minus "none".
- Inclusion–exclusion: $|A \cup B| = |A| + |B| - |A \cap B|$.

Figure: Counting

Four labelled independent draws, five equally likely symbols

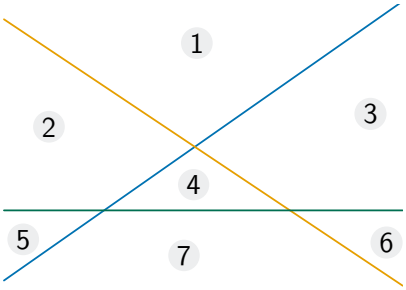
Event	Count model	Count
ordered strings	5 x 5 x 5 x 5	625
all distinct	5 x 4 x 3 x 2	120
at least one repeat	625 - 120	505

- Count “at least one repeat” as the complement of “all distinct”
- Match each part of the diagram to the text.

Recurrences

- A **recurrence** gives each term from earlier ones. Find the pattern of differences, or solve directly.
- n lines in general position divide the plane into R_n regions with $R_n = R_{n-1} + n$, since the nth line crosses $n-1$ others and so is cut into n pieces, each splitting a region. So $R_n = 1 + \dfrac{n(n+1)}{2}$.
- Linear recurrences like $a_n = a_{n-1} + a_{n-2}$ are solved with a characteristic equation, as for differential equations.

Figure: Recurrences



line k crosses $k-1$ lines
and adds k regions:
$1+1+2+3=7$

parallels or three lines
through one point break this

- n lines in general position: 1 + n(n+1)/2 regions.
- Match each part of the diagram to the text.

Probability

- **Mutually exclusive** events cannot both happen: $P(A \cup B) = P(A) + P(B)$.
- Independent events do not affect each other: $P(A \cap B) = P(A)P(B)$. Mutually exclusive events with positive probabilities are never independent.
- Conditional probability: $P(A \mid B) = P(A \cap B)/P(B)$.

Probability (2)

- Expectation is linear, $E[X+Y] = E[X] + E[Y]$, even when X and Y are dependent.
- Binomial: $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$, with mean np and variance $np(1-p)$.

Numerical methods

- Bisection halves an interval that contains a sign change; it always converges, slowly.
- Newton's method: $x_{n+1} = x_n - f(x_n)/f'(x_n)$. Near a simple root the number of correct digits roughly doubles each step.
- The trapezoid rule's error shrinks like h^2; Simpson's rule's like h^4. Simpson's rule is exact for cubics.

Worked example: repeated symbols

- Ordered strings: $5^4 = 625$. All distinct: $5 \times 4 \times 3 \times 2 = 120$.
- $P(\text{repeat}) = 1 - 120/625 = 505/625 = 101/125$.
- Unordered selections of four distinct symbols number $\binom{5}{4} = 5$; each occurs in $4! = 24$ orders. Divide by $4!$ only when order does not matter in both the count and the total.

Worked example: Newton's method

- $x_1 = 1 - (1-2)/2 = 1.5$.
- $x_2 = 1.5 - (2.25-2)/3 = 1.41667$.
- $x_3 = 1.414216$, against the true 1.414214: the digits double each step.

Common errors

- Treating **mutually exclusive** events as independent.
- Adding probabilities of events that can happen together.
- Dividing by $k!$ for an ordered count.

Common errors (2)

- Forgetting that linearity of expectation needs no independence.
- Using Newton's method where f' is zero or near zero.

Key terms

- **recurrence**: A rule giving each term of a sequence from earlier terms.
- **mutually exclusive**: Describing events that cannot happen together.

Check yourself

- How many regions do 5 lines in general position make?
- A fair coin is tossed 10 times. What is the expected number of heads, and the variance?
- What is the probability of at least one six in three rolls of a die?