T.1 · Real analysis and topology

Original classroom teaching; not an official form or score predictor

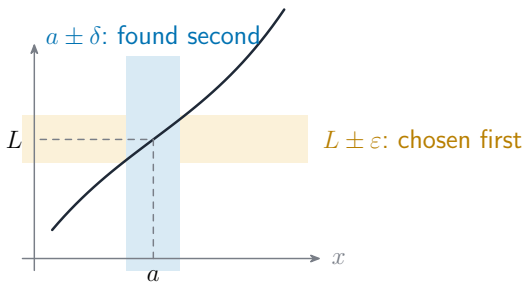
Real analysis and topology

- Use the epsilon–delta definitions of limits and continuity.
- Work with the **supremum** 上确界 and the completeness of the real numbers.
- Decide whether a set is **compact** 紧致的, connected or complete.
- Use the theorems those properties allow, and know when they fail.

Epsilon and delta

- The order matters: ε is given first, and δ may depend on it.
- Continuity at a is the same statement with $L = f(a)$ and the condition $|x - a| < \delta$.
- Uniform continuity asks for one δ that works for every point at once.

Figure: Epsilon and delta



- For every epsilon there is a delta: order matters.
- Read the axes and units first.

Completeness

- Every nonempty set of reals that is bounded above has a least upper bound, its supremum. The rationals lack this: $\{x \in \mathbb{Q} : x^2 < 2\}$ has no rational supremum.
- Monotone convergence: a bounded monotone sequence converges.
- Bolzano–Weierstrass: every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.
- A Cauchy sequence is one whose terms get arbitrarily close to each other; a space is complete when every Cauchy sequence converges in it.

Compactness

- A set is compact if every open cover has a finite subcover.
- Heine–Borel: in $\mathbb{R}^n$, compact means closed and bounded. This equivalence does not hold in every metric space.
- A continuous function on a compact set is bounded, attains its maximum and minimum, and is uniformly continuous.
- $f(x) = x$ on $(0, 1)$ is bounded but never reaches its supremum 1, because $(0, 1)$ is not compact.

Connectedness

- A set is connected if it cannot be split into two disjoint nonempty open pieces. In $\mathbb{R}$ the connected sets are exactly the intervals.
- Continuous images of connected sets are connected; this is the intermediate value theorem.
- Continuous images of compact sets are compact. Completeness, by contrast, is not preserved by continuous maps.

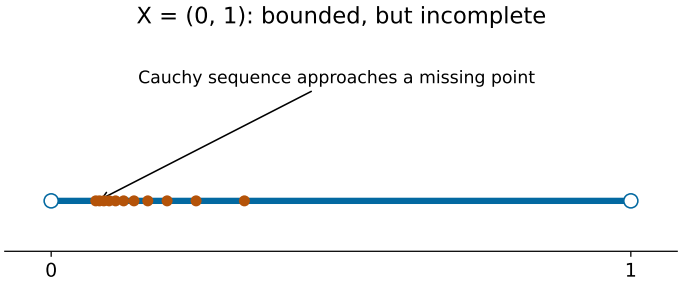
Worked example: epsilon–delta

- Given $\varepsilon > 0$, choose $\delta = \min(1, \varepsilon/5)$.
- If $|x-2| < \delta$, then $1 < x < 3$, so $|x+2| < 5$, and $|x^2 - 4| = |x-2|\,|x+2| < 5\delta \le \varepsilon$.
- On $[1,3]$, which is closed and bounded, f attains a minimum: $f(1) = 1$.

Worked example: an open interval as a space

- X is bounded, and closed in itself (every space is closed in itself).
- The Cauchy sequence $1/(n+2)$ has no limit in X, so X is not complete, and so not compact.
- The cover $\{(1/n, 1) : n \geq 2\}$ has no finite subcover. Heine–Borel does not apply, because X is not closed as a subset of $\mathbb{R}$.

Figure: Worked example: an open interval as a space



- (0, 1) is bounded but not complete: a Cauchy sequence can approach a point outside it
- Match each part of the diagram to the text.

Common errors

- Choosing δ before ε, or letting δ depend on x.
- Using "closed and bounded" for compactness outside $\mathbb{R}^n$.
- Assuming a bounded continuous function attains its supremum on any set.

Common errors (2)

- Confusing complete with compact.
- Thinking the image of a closed set under a continuous map is closed.

Key terms

- **supremum**: The least upper bound of a set.
- **compact**: Having the property that every open cover has a finite subcover.

Check yourself

- What is the supremum of $\{1 - 1/n : n \geq 1\}$? Is it attained?
- Is $[0, \infty)$ compact?
- A continuous function maps $[0, 1]$ onto a set S. Must S be an interval?