

## T.1 · Real analysis and topology

Original classroom teaching; not an official form or score predictor

# 1. Scope and prerequisites

## Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

## 2. Scope and prerequisites

### Scope and prerequisites

Prerequisites: Quantified statements, inequalities, sequences and continuous functions.

### 3. Scope and prerequisites

#### Scope and prerequisites

Apply sequence and function limit definitions Distinguish compactness, connectedness and completeness Use metric-space and elementary topological reasoning

## 4. Scope and prerequisites

### Scope and prerequisites

compact : Every open cover has a finite subcover.

## 5. Scope and prerequisites

### Scope and prerequisites

supremum : The least upper bound of a set.

## 6. Choose and justify a method

### Choose and justify a method

An epsilon–delta statement controls all sufficiently close inputs. The quantifier order matters.

## 7. Choose and justify a method

### Choose and justify a method

In real Euclidean space, closed and bounded sets are compact. Do not apply this equivalence to every metric space.

## 8. Choose and justify a method

### Choose and justify a method

Continuous images of compact sets are compact, so a real continuous function on a compact domain attains extrema.

## 9. Choose and justify a method

### Choose and justify a method

Connectedness rules out a separation into disjoint nonempty open parts. Continuity preserves connectedness; completeness is a separate property.

## 10. Worked reasoning

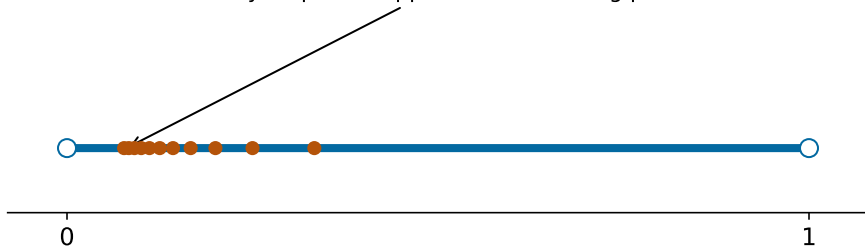
### Worked reasoning

$f(x)=x$  on  $(0,1)$  is continuous and bounded but never equals its supremum 1. On  $[0,1]$ , the same function attains its maximum at 1. The missing endpoint explains why the compact-domain theorem does not apply to the first case.

## Figure 11: Worked reasoning

$X = (0, 1)$ : bounded, but incomplete

Cauchy sequence approaches a missing point



Check the labelled values and conditions.

## 12. Conditions and counterexamples

### Conditions and counterexamples

A theorem's conclusion cannot be used before its hypotheses have been checked.

## 13. Try independently

### Try independently

Prove continuity of  $f(x) = x^2$  at 2 using an explicit epsilon–delta choice. Explain separately why it attains a minimum on  $[1, 3]$  .

## 14. Reasoning and check

### Reasoning and check

Given  $\epsilon > 0$ , choose  $\delta = \min(1, \epsilon/5)$ . If  $|x - 2| < \delta$ , then  $1 < x < 3$ , so  $|x + 2| < 5$ .  $|x^2 - 4| = |x - 2||x + 2| < 5\delta \leq \epsilon$ . The choice depends on epsilon, not on the later input  $x$ . The interval  $[1, 3]$  is nonempty and compact in the ordinary real metric. Continuity gives attained extrema. Here  $x^2 \geq 1$ , with equality at 1, so the minimum is 1.

## 15. Try independently

### Try independently

In the metric space  $X = (0, 1)$  with ordinary distance, is  $X$  bounded? Is it closed in itself? Is it compact or complete? Explain why “closed and bounded implies compact” cannot be used here.

## 16. Reasoning and check

### Reasoning and check

$X$  is bounded and closed relative to itself, since its complement in  $X$  is empty. The Cauchy sequence  $1/(n+2)$  has no limit in  $X$ , so  $X$  is not complete and therefore not compact as a metric space. Directly, the relative open cover  $\{(1/n, 1) : n \geq 2\}$  covers  $X$  but has no finite subcover. Heine–Borel requires closed and bounded as a subset of Euclidean space, not merely closed in an arbitrary chosen space.  $X$  is not closed in  $\mathbb{R}$ . No theorem hypothesis may change its ambient space halfway through the argument.

## Check yourself

Reconstruct the method without looking back

In the metric space  $X = (0, 1)$  with ordinary distance, is  $X$  bounded? Is it closed in itself? Is it compact or complete? Explain why “closed and bounded implies compact” cannot be used here.