A.1 · Linear algebra

Original classroom teaching; not an official form or score predictor

Linear algebra

- Solve linear systems and find the **rank** 秩 and **kernel** 核 of a matrix.
- Use the rank–nullity theorem.
- Use determinants and their properties.
- Find **eigenvalues** 特征值 and eigenvectors, and decide when a matrix is diagonalisable.

Systems and rank

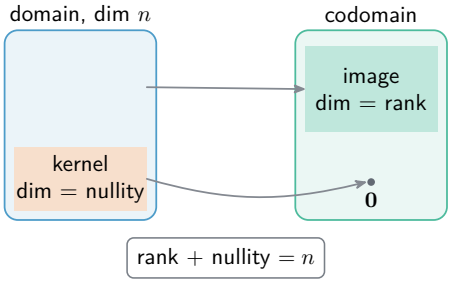
- Row reduction solves $Ax = b$. The system has a solution exactly when b is in the column space of A.
- The rank is the dimension of the image (column space), equal to the number of pivots.
- The kernel (null space) is the set of solutions of $Ax = 0$; its dimension is the nullity.
- All solutions of $Ax = b$ are one particular solution plus the kernel.

Rank–nullity

- A 5×7 matrix of rank 3 has nullity $7 - 3 = 4$.
- A map from $\mathbb{R}^3$ to $\mathbb{R}^2$ can never be one-to-one.

$$\operatorname{rank} T + \operatorname{nullity} T = \dim V$$

Figure: Rank–nullity



- Rank plus nullity equals the domain's dimension.
- Match each part of the diagram to the text.

Determinants

- $\det(AB) = \det A \det B$, $\det A^T = \det A$, and $\det(cA) = c^n \det A$ for an $n \times n$ matrix.
- A is invertible exactly when $\det A \neq 0$.
- $|\det A|$ is the factor by which A scales areas or volumes.
- Swapping two rows changes the sign; adding a multiple of one row to another changes nothing.

Eigenvalues

- The sum of the eigenvalues is the trace; their product is the determinant.
- A triangular matrix has its diagonal entries as eigenvalues.
- A is diagonalisable when it has n independent eigenvectors. Distinct eigenvalues guarantee this; a repeated eigenvalue needs an eigenspace of full dimension.
- Real symmetric matrices are always diagonalisable, with real eigenvalues and orthogonal eigenvectors.

Figure: Eigenvalues

A repeated eigenvalue can have different eigenspace dimensions

Matrix	Eigenvalue	Dimension	Diagonalizable?
[[1, 1], [0, 1]]	1, twice	1	No
[[1, 0], [0, 1]]	1, twice	2	Yes

- A repeated eigenvalue gives a diagonalizable matrix only when its eigenspace has full dimension
- Match each part of the diagram to the text.

Worked example: kernel and image

- The second row is twice the first, so the rank is 1, and every output has the form $(s, 2s)$.
- Kernel: $x = -2y - z$, so $\ker T = \operatorname{span}\{(-2, 1, 0), (-1, 0, 1)\}$, nullity 2. Check: $1 + 2 = 3$.
- $T(v) = (3, 6)$ has solutions $(3 - 2a - b, a, b)$; $T(v) = (3, 7)$ has none, since $7 \neq 2 \times 3$.

Worked example: eigenvalues

- $\det(A - \lambda I) = (2 - \lambda)^2 - 1 = 0$ gives $\lambda = 3$ and $\lambda = 1$. Check: trace 4, determinant 3.
- Eigenvectors: $(1, 1)$ for 3 and $(1, -1)$ for 1. They are orthogonal, as for any symmetric matrix.
- A^{10} has eigenvalues 3^{10} and 1, with the same eigenvectors.

Common errors

- Treating a repeated eigenvalue as proof that a matrix is not diagonalisable (the identity is diagonal).
- Writing $\det(cA) = c\det A$.
- Forgetting that the general solution includes the whole kernel.

Common errors (2)

- Mixing up the number of rows and columns in rank–nullity: use the dimension of the domain.
- Thinking $\det(A + B) = \det A + \det B$.

Key terms

- **eigenvector**: A nonzero vector that a matrix only stretches: $Av = \lambda v$.
- **rank**: The dimension of the image of a linear map.
- **kernel**: The set of vectors that a linear map sends to zero.
- **eigenvalues**: The scalars λ for which $Av = \lambda v$ has a nonzero solution.

Check yourself

- A 3×3 matrix has $\det A = 5$. What is $\det(2A)$?
- Is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ diagonalisable?
- A 4×6 matrix has rank 4. What is the dimension of its kernel?