C.2 · Multivariable calculus and vector analysis

Original classroom teaching; not an official form or score predictor

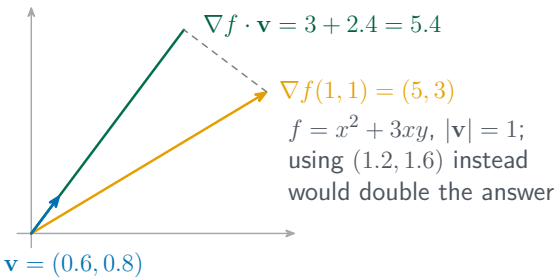
Multivariable calculus and vector analysis

- Compute partial derivatives, the **gradient** 梯度 and directional derivatives.
- Find tangent planes and use the multivariable chain rule.
- Evaluate double and triple integrals, including changes of coordinates with the **Jacobian** 雅可比行列式.
- Evaluate line integrals and use Green's theorem.

The gradient

- The directional derivative in the direction of a unit vector u is $D_u f = \nabla f \cdot u$. The direction must be a unit vector.
- ∇f points in the direction of fastest increase, and its length $|\nabla f|$ is that fastest rate.
- ∇f is perpendicular to the level curves (contours) of f.

Figure: The gradient

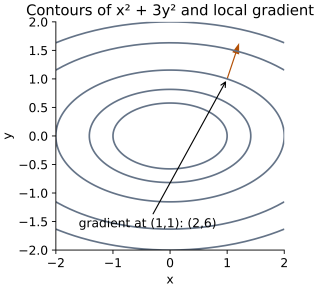


- Directional derivative: gradient dotted with a unit vector.
- Match each part of the diagram to the text.

Tangent planes and the chain rule

- The tangent plane to $z = f(x, y)$ at (a, b) is $z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$.
- Chain rule: if x and y depend on t, then $\dfrac{df}{dt} = f_x \dfrac{dx}{dt} + f_y \dfrac{dy}{dt}$.
- If $F(x, y, z) = 0$ defines a surface, ∇F is normal to it.

Figure: Tangent planes and the chain rule



- The gradient (2, 6) at (1, 1) is perpendicular to the contour and points uphill
- Match each part of the diagram to the text.

Multiple integrals

- Fubini's theorem: for a continuous function on a rectangle, the order of integration does not matter.
- When changing coordinates, multiply by the absolute value of the **Jacobian** determinant.

$dA = r\,dr\,d\theta$ (polar), $dV = r\,dr\,d\theta\,dz$ (cylindrical), $dV = \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$ (spherical)

Line integrals and Green's theorem

- The work of a field F along a curve C is $\int_C F \cdot dr$.
- If $F = \nabla f$, the integral depends only on the endpoints: $f(\text{end}) - f(\text{start})$.
- Green's theorem, for a curve C around a region D taken anticlockwise:
  $$\oint_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA.$$

Worked example: a directional derivative

- Normalise: $u = (3/5, 4/5)$.
- $\nabla f = (2x + y, x + 4y)$, which is $(1, -3)$ at $(1, -1)$.
- $D_u f = 1 \times \frac{3}{5} + (-3) \times \frac{4}{5} = -\frac{9}{5}$. The fastest rate is $|\nabla f| = \sqrt{10}$.

Worked example: a polar integral

- In polar coordinates $x^2 + y^2 = r^2$ and $dA = r\,dr\,d\theta$.
- $\displaystyle\int_0^{2\pi}\int_0^{1} r^3\,dr\,d\theta = 2\pi \times \tfrac{1}{4} = \tfrac{\pi}{2}$.
- Forgetting the factor r would give $2\pi/3$, a common wrong option.

Common errors

- Using a direction vector that is not a unit vector.
- Forgetting the **Jacobian** factor r or $\rho^2 \sin\phi$.
- Thinking the gradient is parallel to the level curves.

Common errors (2)

- Using Green's theorem with a clockwise curve without changing the sign.
- Treating a line integral of a non-gradient field as path-independent.

Key terms

- **gradient**: The vector of partial derivatives, pointing in the direction of fastest increase.
- **Jacobian**: The determinant that gives the local area or volume scaling of a coordinate change.

Check yourself

- What is the largest rate of increase of $f = x^2 + 3y^2$ at $(1, 1)$?
- What is the tangent plane to $z = x^2 + 3y^2$ at $(1, 1, 4)$?
- What is $\oint_C x\,dy$ around the unit circle, anticlockwise?