

## C.2 · Multivariable calculus and vector analysis

Original classroom teaching; not an official form or score predictor

# 1. Scope and prerequisites

## Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

## 2. Scope and prerequisites

### Scope and prerequisites

Prerequisites: Single-variable derivatives, dot products and iterated integration.

### 3. Scope and prerequisites

#### Scope and prerequisites

Compute partial derivatives, gradients and directional derivatives Use multiple integrals and coordinate changes Use derivative conditions to distinguish planes from curved surfaces

## 4. Scope and prerequisites

### Scope and prerequisites

gradient : Vector of partial derivatives.

## 5. Scope and prerequisites

### Scope and prerequisites

Jacobian : The local area or volume scaling in a coordinate change.

## 6. Choose and justify a method

### Choose and justify a method

A partial derivative changes one coordinate while fixing the others. For  $f=x^2+3xy$ ,  $f_x=2x+3y$  and  $f_y=3x$ . The gradient collects these derivatives; if  $f$  is differentiable, the directional derivative along a unit vector  $v$  is  $\text{grad } f \cdot v$ . Normalise the direction before taking this dot product. A direction vector of length two would double the answer if used without normalisation.

## 7. Choose and justify a method

### Choose and justify a method

Differentiability means a valid linear approximation, not merely the existence of some partial derivatives at one point. Continuous first partials in a neighbourhood are a sufficient condition. For a composition  $f(x(t), y(t))$ , the chain rule gives  $f_x x' + f_y y'$ . Second derivatives can introduce both direct and mixed terms; a zero mixed partial alone places no restriction on the pure second partials.

## 8. Choose and justify a method

### Choose and justify a method

A multiple integral sums contributions over a specified region. Describe the region before choosing iterated limits; nonrectangular bounds may change when the order changes. In polar coordinates area is  $r \, dr \, d\theta$ , not just  $dr \, d\theta$ . A general coordinate change uses the absolute Jacobian determinant. Sign belongs to oriented vector quantities, while area and volume scaling use a nonnegative factor.

## 9. Choose and justify a method

### Choose and justify a method

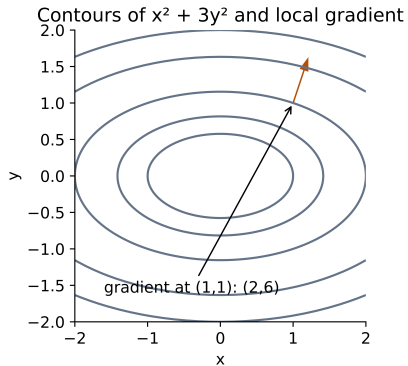
If both first partials of a globally defined function on  $\mathbb{R}^2$  are constant,  $f_x=a$  and  $f_y=b$ , integrating successively gives  $f=ax+by+c$ , a plane. This cannot be inferred from parallel straight level sets alone:  $e^x$  has vertical parallel level lines but a curved graph. Nor do  $f_{xy}=f_{yx}=0$  force a plane:  $x^2+y^2$  is a counterexample. Distinguish first-derivative constancy from absent mixed dependence and from the shape of selected level sets.

## 10. Worked reasoning

### Worked reasoning

For  $f(x,y)=x^2+3y^2$ , gradient  $\nabla f=(2x,6y)$ . At  $(1,1)$  this is  $(2,6)$ . In direction  $(3,4)$ , the unit vector is  $(3/5,4/5)$ , giving directional derivative  $2 \cdot 3/5 + 6 \cdot 4/5 = 6$ .

Figure 11: Worked reasoning



Check the labelled values and conditions.

## 12. Conditions and counterexamples

### Conditions and counterexamples

A non-unit direction vector gives a scaled directional rate, not the derivative per unit distance.

## 13. Try independently

### Try independently

For  $f(x, y) = x^2 + xy + 2y^2$ , find the directional derivative at  $(1, -1)$  towards  $(3, 4)$ .  
Evaluate  $\iint_D (x^2 + y^2) dA$  for the unit disk  $D$ .

## 14. Reasoning and check

### Reasoning and check

The polynomial is differentiable. Normalise the direction:  $u = (3/5, 4/5)$ . The gradient is  $(2x + y, x + 4y)$ , hence  $(1, -3)$  at the point.

$D_u f(1, -1) = \nabla f(1, -1) \cdot u = 1 \cdot 3/5 - 3 \cdot 4/5 = -9/5$ . Polar coordinates give  $x^2 + y^2 = r^2$  and  $dA = r dr d\theta$  :  $I = \int_0^{2\pi} \int_0^1 r^3 dr d\theta = \pi/2$ .

## 15. Try independently

### Try independently

Define  $g(x, y) = xy/\sqrt{x^2 + y^2}$  away from the origin and  $g(0, 0) = 0$ . Both partial derivatives at the origin are zero. Is  $g$  differentiable there? Is it continuous?

## 16. Reasoning and check

### Reasoning and check

Along each axis  $g = 0$ , so both partials are zero. Continuity follows from  $|xy| \leq (x^2 + y^2)/2$ , which gives  $|g(x, y)| \leq \sqrt{x^2 + y^2}/2 \rightarrow 0$ . If differentiable, its linear derivative would be zero. Along  $x = y = t \neq 0$ ,  $\frac{|g(t, t)|}{\sqrt{t^2 + t^2}} = \frac{1}{2}$ . The required remainder ratio does not tend to zero. Thus continuity and existing partial derivatives do not establish differentiability.

## Check yourself

Reconstruct the method without looking back

Define  $g(x, y) = xy/\sqrt{x^2 + y^2}$  away from the origin and  $g(0, 0) = 0$ . Both partial derivatives at the origin are zero. Is  $g$  differentiable there? Is it continuous?