C.1 · Single-variable calculus and applications

Original classroom teaching; not an official form or score predictor

Single-variable calculus and applications

- Find limits and judge **continuity 连续性** and differentiability.
- Use the mean value theorem and the intermediate value theorem.
- Apply the **fundamental theorem of calculus 微积分基本定理**, including variable limits.

Single-variable calculus and applications (2)

- Solve optimisation problems and use L'Hôpital's rule.
- Judge the **convergence** 收敛 of improper integrals and series.

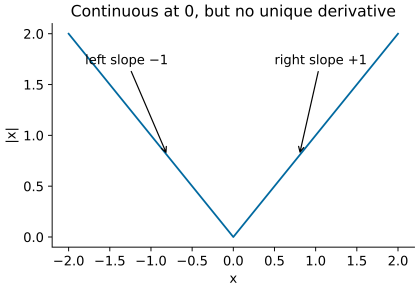
Limits and continuity

- $\dfrac{x^2-1}{x-1}$ is undefined at 1, but its limit there is 2: a removable discontinuity.
- A jump (different one-sided limits) or an infinite limit cannot be removed.

Derivatives

- Differentiable implies continuous, but not the reverse: $|x|$ is continuous at 0 with one-sided slopes -1 and $+1$, so it has no derivative there.
- $x|x|$ is differentiable at 0, with derivative 0.
- Mean value theorem: if f is continuous on $[a, b]$ and differentiable on (a, b), some c has $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.
- Intermediate value theorem: a continuous function on $[a, b]$ takes every value between $f(a)$ and $f(b)$.

Figure: Derivatives



- $|x|$ is continuous at 0, but the one-sided slopes -1 and $+1$ disagree, so there is no derivative
- Read the axes and units first.

The fundamental theorem

- The second part: $\int_a^b f(x)\,dx = F(b) - F(a)$ for any antiderivative F.
- If f is continuous, then $\dfrac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f(v(x))\,v'(x) - f(u(x))\,u'(x)$$

Applications

- Optimisation: find critical points where $f' = 0$ or f' is undefined, then compare them with the endpoints.
- L'Hôpital's rule: for the forms $0/0$ or ∞/∞, $\lim f/g = \lim f'/g'$, if the second limit exists. Check the form first.
- Techniques of integration: substitution, integration by parts $\int u\,dv = uv - \int v\,du$, and partial fractions.
- $\displaystyle\int_1^\infty x^{-p}\,dx$ converges exactly when $p > 1$; $\sum n^{-p}$ behaves the same way.

Worked example: a variable limit

- The integrand is continuous, so the theorem applies: $F'(x) = (1 + x^4)(2x) - (1 + x^2)$.
- $F'(2) = 17 \times 4 - 5 = 63$.
- Check: an antiderivative is $t + t^3/3$, so $F = x^2 + x^6/3 - x - x^3/3$ and $F' = 2x + 2x^5 - 1 - x^2$, which is 63 at 2.

Worked example: an open box

- $V = x(12-2x)^2$ on $0 \le x \le 6$.
- $V' = (12-2x)^2 - 4x(12-2x) = (12-2x)(12-6x)$, which is zero at $x = 2$ or $x = 6$.
- $V(2) = 2 \times 64 = 128$ cm^3; the endpoints give 0. A limit by L'Hôpital, for comparison: $\lim_{x\to 0} \dfrac{\sin x - x}{x^3} = \lim \dfrac{\cos x - 1}{3x^2} = \lim \dfrac{-\sin x}{6x} = -\dfrac{1}{6}$.

Common errors

- Assuming **continuity** implies differentiability.
- Forgetting the chain-rule factor at a variable limit, or the lower limit's term.
- Applying L'Hôpital's rule to a form that is not $0/0$ or ∞/∞.

Common errors (2)

- Checking critical points but not endpoints.
- Thinking $\int_1^\infty x^{-1}\,dx$ converges.

Key terms

- **continuity**: The agreement of a function's value with its limit at a point.
- **fundamental theorem of calculus**: The result linking differentiation and integration: the derivative of an accumulated integral is the integrand.
- **convergence**: Approach to a finite limiting value.

Check yourself

- What is $\dfrac{d}{dx}\displaystyle\int_0^{x^3} e^{t^2}\,dt$?
- What is $\lim_{x\to 0}\dfrac{1-\cos x}{x^2}$?
- Does $\displaystyle\int_1^{\infty} x^{-1/2}\,dx$ converge?