Course Companion

GRE Subject Test · GRE Mathematics

Every topic handout in one volume

全部专题讲义合集

exercises.lol

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C.1 · Single-variable calculus and applications

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Single-variable calculus and applications

In 1684 Gottfried Leibniz published a six-page paper with rules for differentiating sums, products and quotients, using the dx notation still used today. Isaac Newton had developed the same ideas twenty years earlier but had not published them. A bitter dispute followed over who invented calculus. In 1712 the Royal Society, of which Newton was president, issued a report finding in his favour; Newton had secretly written most of it himself. Today both are credited, and Leibniz's notation won.

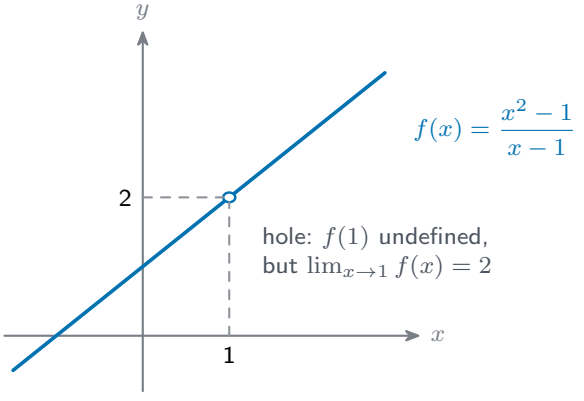
GRE Mathematics calculus questions ask you to:

- Find limits and judge **continuity** 连续性 and differentiability.
- Use the mean value theorem and the intermediate value theorem.
- Apply the **fundamental theorem of calculus** 微积分基本定理, including variable limits.
- Solve optimisation problems and use L'Hôpital's rule.
- Judge the **convergence** 收敛 of improper integrals and series.

Limits and continuity

f is continuous at a when $\lim_{x\to a} f(x) = f(a)$: the limit exists, the value exists, and they agree.

- $\dfrac{x^2-1}{x-1}$ is undefined at 1, but its limit there is 2: a removable discontinuity.
- A jump (different one-sided limits) or an infinite limit cannot be removed.

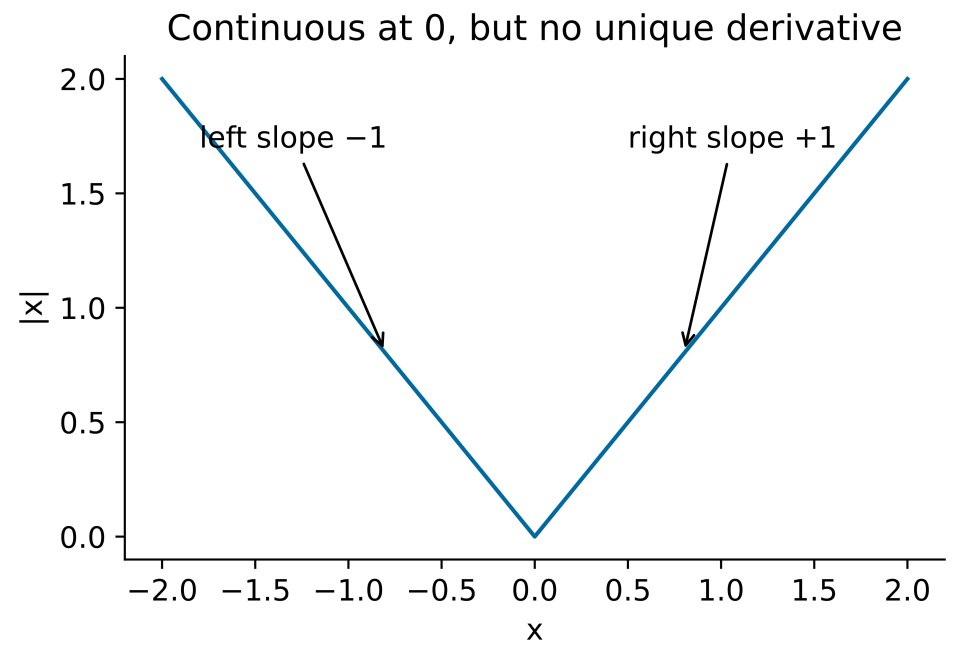


A removable discontinuity: the limit exists where the function does not.

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Derivatives

- Differentiable implies continuous, but not the reverse: $|x|$ is continuous at 0 with one-sided slopes -1 and $+1$, so it has no derivative there.
- $x|x|$ is differentiable at 0, with derivative 0.
- Mean value theorem: if f is continuous on $[a, b]$ and differentiable on (a, b), some c has $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.
- Intermediate value theorem: a continuous function on $[a, b]$ takes every value between $f(a)$ and $f(b)$.



|x| is continuous at 0, but the one-sided slopes −1 and +1 disagree, so there is no derivative

The fundamental theorem

If f is continuous, then $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$. With variable limits, use the chain rule at each limit:

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f(v(x))\,v'(x) - f(u(x))\,u'(x)$$

- The second part: $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$ for any antiderivative F.

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Applications

- Optimisation: find critical points where $f' = 0$ or f' is undefined, then compare them with the endpoints.
- L'Hôpital's rule: for the forms $0/0$ or ∞/∞, $\lim f/g = \lim f'/g'$, if the second limit exists. Check the form first.
- Techniques of integration: substitution, integration by parts $\int u\,dv = uv - \int v\,du$, and partial fractions.
- $\displaystyle\int_1^\infty x^{-p}\,dx$ converges exactly when $p > 1$; $\sum n^{-p}$ behaves the same way.

Worked example: a variable limit

Let $F(x) = \displaystyle\int_x^{x^2} (1 + t^2)\,dt$. Find $F'(2)$.

- The integrand is continuous, so the theorem applies: $F'(x) = (1 + x^4)(2x) - (1 + x^2)$.
- $F'(2) = 17 \times 4 - 5 = 63$.
- Check: an antiderivative is $t + t^3/3$, so $F = x^2 + x^6/3 - x - x^3/3$ and $F' = 2x + 2x^5 - 1 - x^2$, which is 63 at 2.

Worked example: an open box

Cut squares of side x from the corners of a 12 cm square sheet and fold up the sides. Maximise the volume.

- $V = x(12 - 2x)^2$ on $0 \le x \le 6$.
- $V' = (12 - 2x)^2 - 4x(12 - 2x) = (12 - 2x)(12 - 6x)$, which is zero at $x = 2$ or $x = 6$.
- $V(2) = 2 \times 64 = 128$ cm^3; the endpoints give 0. A limit by L'Hôpital, for comparison: $\lim_{x\to 0} \dfrac{\sin x - x}{x^3} = \lim \dfrac{\cos x - 1}{3x^2} = \lim \dfrac{-\sin x}{6x} = -\dfrac{1}{6}$.

Common errors

- Assuming continuity implies differentiability.
- Forgetting the chain-rule factor at a variable limit, or the lower limit's term.
- Applying L'Hôpital's rule to a form that is not $0/0$ or ∞/∞.
- Checking critical points but not endpoints.
- Thinking $\int_1^\infty x^{-1}\,dx$ converges.

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Key terms

continuity: The agreement of a function's value with its limit at a point.

fundamental theorem of calculus: The result linking differentiation and integration: the derivative of an accumulated integral is the integrand.

convergence: Approach to a finite limiting value.

Check yourself

- What is $\dfrac{d}{dx}\displaystyle\int_0^{x^3} e^{t^2}\,dt$?
- What is $\lim_{x\to 0}\dfrac{1-\cos x}{x^2}$?
- Does $\displaystyle\int_1^{\infty} x^{-1/2}\,dx$ converge?

Answers: $3x^2e^{x^6}$. $\frac{1}{2}$. No, since $p=\frac{1}{2}\le 1$.

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C.2 · Multivariable calculus and vector analysis

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Multivariable calculus and vector analysis

In 1881 Josiah Willard Gibbs, a physics professor at Yale, printed a short pamphlet for his students called *Elements of Vector Analysis*. He replaced the complicated quaternions of the time with simpler dot and cross products, the gradient, divergence and curl. In England, the self-taught engineer Oliver Heaviside reached almost the same system independently. Their notation is the one in every multivariable calculus course today.

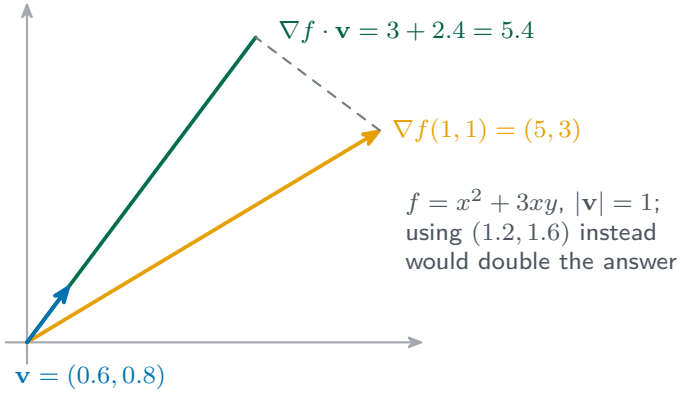
GRE Mathematics multivariable questions ask you to:

- Compute partial derivatives, the **gradient** 梯度 and directional derivatives.
- Find tangent planes and use the multivariable chain rule.
- Evaluate double and triple integrals, including changes of coordinates with the **Jacobian** 雅可比行列式.
- Evaluate line integrals and use Green's theorem.

The gradient

$\nabla f = (f_x, f_y)$ collects the partial derivatives.

- The directional derivative in the direction of a unit vector u is $D_u f = \nabla f \cdot u$. The direction must be a unit vector.
- ∇f points in the direction of fastest increase, and its length $|\nabla f|$ is that fastest rate.
- ∇f is perpendicular to the level curves (contours) of f.

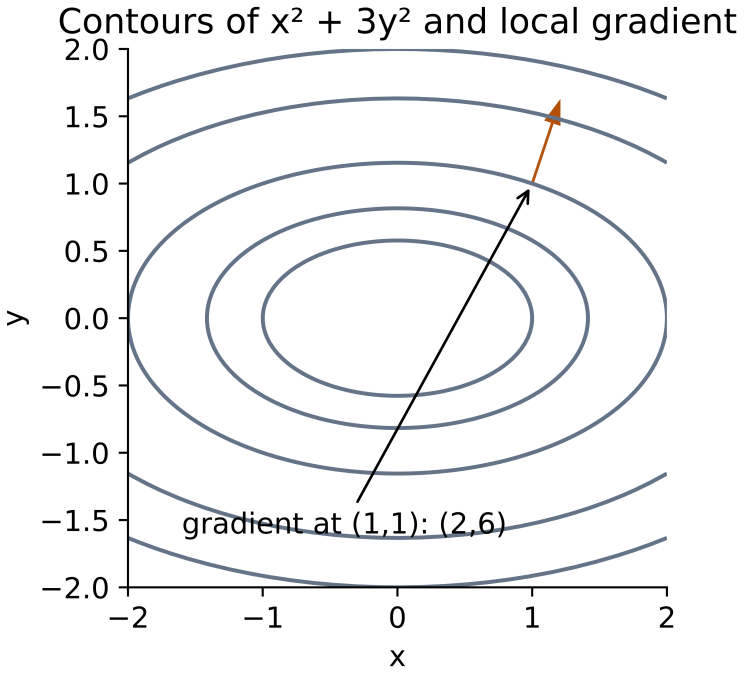


Directional derivative: gradient dotted with a unit vector.

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Tangent planes and the chain rule

- The tangent plane to $z = f(x, y)$ at (a, b) is $z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$.
- Chain rule: if x and y depend on t, then $\dfrac{df}{dt} = f_x \dfrac{dx}{dt} + f_y \dfrac{dy}{dt}$.
- If $F(x, y, z) = 0$ defines a surface, ∇F is normal to it.



The gradient (2, 6) at (1, 1) is perpendicular to the contour and points uphill

Multiple integrals

- Fubini's theorem: for a continuous function on a rectangle, the order of integration does not matter.
- When changing coordinates, multiply by the absolute value of the Jacobian determinant.

$dA = r\,dr\,d\theta$ (polar), $dV = r\,dr\,d\theta\,dz$ (cylindrical), $dV = \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$ (spherical)

Line integrals and Green's theorem

- The work of a field F along a curve C is $\int_C F \cdot dr$.
- If $F = \nabla f$, the integral depends only on the endpoints: $f(\text{end}) - f(\text{start})$.
- Green's theorem, for a curve C around a region D taken anticlockwise: $\oint_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$.

Worked example: a directional derivative

Let $f(x, y) = x^2 + xy + 2y^2$. Find the rate of change at $(1, -1)$ toward the direction $(3, 4)$.

- Normalise: $u = (3/5, 4/5)$.
- $\nabla f = (2x + y, x + 4y)$, which is $(1, -3)$ at $(1, -1)$.
- $D_u f = 1 \times \frac{3}{5} + (-3) \times \frac{4}{5} = -\frac{9}{5}$. The fastest rate is $|\nabla f| = \sqrt{10}$.

Worked example: a polar integral

Evaluate $\iint_D (x^2 + y^2)\,dA$ over the unit disc.

- In polar coordinates $x^2 + y^2 = r^2$ and $dA = r\,dr\,d\theta$.
- $\int_0^{2\pi} \int_0^1 r^3\,dr\,d\theta = 2\pi \times \frac{1}{4} = \frac{\pi}{2}$.
- Forgetting the factor r would give $2\pi/3$, a common wrong option.

Common errors

- Using a direction vector that is not a unit vector.
- Forgetting the Jacobian factor r or $\rho^2 \sin\phi$.
- Thinking the gradient is parallel to the level curves.
- Using Green's theorem with a clockwise curve without changing the sign.
- Treating a line integral of a non-gradient field as path-independent.

Key terms

gradient: The vector of partial derivatives, pointing in the direction of fastest increase.

Jacobian: The determinant that gives the local area or volume scaling of a coordinate change.

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Check yourself

- What is the largest rate of increase of $f = x^2 + 3y^2$ at $(1, 1)$?
- What is the tangent plane to $z = x^2 + 3y^2$ at $(1, 1, 4)$?
- What is $\oint_C x\,dy$ around the unit circle, anticlockwise?

Answers: $|(2, 6)| = 2\sqrt{10}$. $z = 4 + 2(x - 1) + 6(y - 1)$. π, the area of the disc.

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A.1 · Linear algebra

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Linear algebra

In 1998 two Stanford graduate students, Sergey Brin and Larry Page, described a way to rank web pages. A page is important if important pages link to it. Written as equations, the importance scores of all pages form an **eigenvector** 特征向量 of a huge matrix of links, with eigenvalue 1. Their search engine, Google, was built on computing that eigenvector for millions of pages. Linear algebra, once a topic for solving small systems by hand, now runs much of the modern world.

GRE Mathematics linear algebra questions ask you to:

- Solve linear systems and find the **rank** 秩 and **kernel** 核 of a matrix.
- Use the rank–nullity theorem.
- Use determinants and their properties.
- Find **eigenvalues** 特征值 and eigenvectors, and decide when a matrix is diagonalisable.

Systems and rank

- Row reduction solves $Ax = b$. The system has a solution exactly when b is in the column space of A.
- The rank is the dimension of the image (column space), equal to the number of pivots.
- The kernel (null space) is the set of solutions of $Ax = 0$; its dimension is the nullity.
- All solutions of $Ax = b$ are one particular solution plus the kernel.

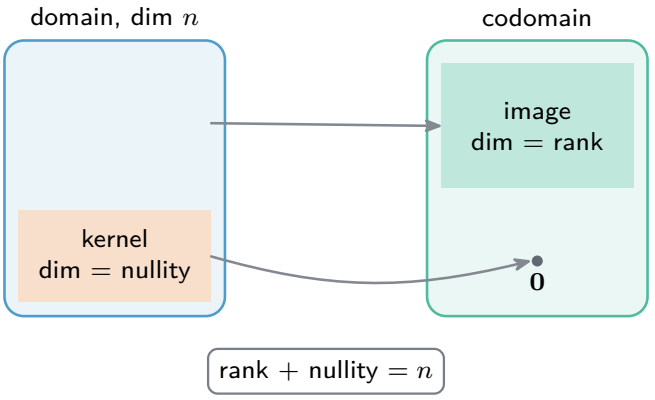
Rank–nullity

For a linear map $T : V \to W$ with V finite-dimensional:

$$\operatorname{rank} T + \operatorname{nullity} T = \dim V$$

- A 5×7 matrix of rank 3 has nullity $7 - 3 = 4$.
- A map from $\mathbb{R}^3$ to $\mathbb{R}^2$ can never be one-to-one.

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Rank plus nullity equals the domain's dimension.

Determinants

- $\det(AB) = \det A \det B$, $\det A^T = \det A$, and $\det(cA) = c^n \det A$ for an $n \times n$ matrix.
- A is invertible exactly when $\det A \neq 0$.
- $|\det A|$ is the factor by which A scales areas or volumes.
- Swapping two rows changes the sign; adding a multiple of one row to another changes nothing.

Eigenvalues

$Av = \lambda v$ with $v \neq 0$. The eigenvalues are the roots of $\det(A - \lambda I) = 0$.

- The sum of the eigenvalues is the trace; their product is the determinant.
- A triangular matrix has its diagonal entries as eigenvalues.
- A is diagonalisable when it has n independent eigenvectors. Distinct eigenvalues guarantee this; a repeated eigenvalue needs an eigenspace of full dimension.
- Real symmetric matrices are always diagonalisable, with real eigenvalues and orthogonal eigenvectors.

A repeated eigenvalue can have different eigenspace dimensions

Matrix	Eigenvalue	Dimension	Diagonalizable?
[[1, 1], [0, 1]]	1, twice	1	No
[[1, 0], [0, 1]]	1, twice	2	Yes

A repeated eigenvalue gives a diagonalizable matrix only when its eigenspace has full dimension

Worked example: kernel and image

$T : \mathbb{R}^3 \to \mathbb{R}^2$, $T(x, y, z) = (x + 2y + z,\ 2x + 4y + 2z)$.

- The second row is twice the first, so the rank is 1, and every output has the form $(s, 2s)$.
- Kernel: $x = -2y - z$, so $\ker T = \operatorname{span}\{(-2, 1, 0), (-1, 0, 1)\}$, nullity 2. Check: $1 + 2 = 3$.
- $T(v) = (3, 6)$ has solutions $(3-2a-b, a, b)$; $T(v) = (3, 7)$ has none, since $7 \neq 2\times 3$.

Worked example: eigenvalues

$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

- $\det(A - \lambda I) = (2 - \lambda)^2 - 1 = 0$ gives $\lambda = 3$ and $\lambda = 1$. Check: trace 4, determinant 3.
- Eigenvectors: $(1, 1)$ for 3 and $(1, -1)$ for 1. They are orthogonal, as for any symmetric matrix.
- A^{10} has eigenvalues 3^{10} and 1, with the same eigenvectors.

Common errors

- Treating a repeated eigenvalue as proof that a matrix is not diagonalisable (the identity is diagonal).
- Writing $\det(cA) = c \det A$.
- Forgetting that the general solution includes the whole kernel.

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- Mixing up the number of rows and columns in rank–nullity: use the dimension of the domain.
- Thinking $\det(A+B) = \det A + \det B$.

Key terms

eigenvector: A nonzero vector that a matrix only stretches: $Av = \lambda v$.

rank: The dimension of the image of a linear map.

kernel: The set of vectors that a linear map sends to zero.

eigenvalues: The scalars λ for which $Av = \lambda v$ has a nonzero solution.

Check yourself

- A 3×3 matrix has $\det A = 5$. What is $\det(2A)$?
- Is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ diagonalisable?
- A 4×6 matrix has rank 4. What is the dimension of its kernel?

Answers: $2^3 \times 5 = 40$. No: its only eigenspace is one-dimensional. 2.

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A.2 · Abstract algebra and number theory

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Abstract algebra and number theory

On the night of 29 May 1832 Évariste Galois, aged 20, sat up in Paris writing a letter to a friend, summarising his mathematical discoveries in case he died. The next morning he was shot in a duel, and he died the day after. His ideas, about the symmetries of the roots of polynomial equations, explained why there is no general formula for equations of degree five. They were published fourteen years later and became the foundation of group theory, the study of symmetry.

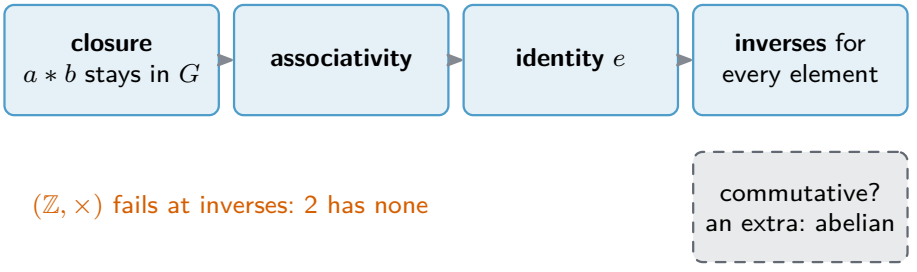
GRE Mathematics algebra questions ask you to:

- Check the group axioms and use Lagrange's theorem.
- Work with cyclic groups and the orders of elements.
- Use a **homomorphism** 同态 and its kernel.
- Tell rings from **fields** 域, and find the units of $\mathbb{Z}/n\mathbb{Z}$.
- Use the Euclidean algorithm and Fermat's little theorem.

Groups

A group is a set with an operation that satisfies four axioms:

- Closure, associativity, an identity element, and an inverse for every element.
- Commutativity is extra: groups that have it are abelian.
- The integers form a group under addition, but not under multiplication: 2 has no integer inverse.



Check every axiom; commutativity is optional.

Subgroups and order

- Lagrange's theorem: in a finite group, the order of every subgroup divides the order of the group. So does the order of every element.
- The converse is false: a group of order 12 (the alternating group A_4) has no subgroup of order 6.
- A cyclic group of order n is generated by one element; in $\mathbb{Z}/n\mathbb{Z}$ the element k has order $n/\gcd(k,n)$.
- Every group of prime order is cyclic.

Homomorphisms

A homomorphism $\phi : G \to H$ satisfies $\phi(ab) = \phi(a)\phi(b)$.

- Its kernel, the elements sent to the identity, is a normal subgroup.
- ϕ is one-to-one exactly when its kernel is only the identity.
- First isomorphism theorem: $G/\ker\phi \cong \operatorname{im}\phi$, so $|G| = |\ker\phi|\,|\operatorname{im}\phi|$ for finite G.

Rings and fields

- A ring has addition and multiplication; a field is a commutative ring in which every nonzero element has a multiplicative inverse.
- In $\mathbb{Z}/n\mathbb{Z}$, a is a unit exactly when $\gcd(a,n) = 1$. There are $\varphi(n)$ units, where φ is Euler's function.
- $\mathbb{Z}/n\mathbb{Z}$ is a field exactly when n is prime.

Same addition rule; different invertibility

Ring	Multiplicative units	Check
mod 5	1, 2, 3, 4	every nonzero residue
mod 6	1, 5	2 times 3 is zero

Mod 5 every nonzero residue is invertible; mod 6 the zero divisors 2 and 3 are not

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Number theory

- The Euclidean algorithm finds $\gcd(a, b)$ by repeated division, and gives integers x, y with $ax + by = \gcd(a, b)$.
- Fermat's little theorem: if p is prime and $p \nmid a$, then $a^{p-1} \equiv 1 \pmod{p}$.
- Euler's theorem: if $\gcd(a, n) = 1$, then $a^{\varphi(n)} \equiv 1 \pmod{n}$.

Worked example: two rings

Compare $\mathbb{Z}/5\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$.

- Both are groups under addition.
- Units mod 5: 1, 2, 3, 4, so $\mathbb{Z}/5\mathbb{Z}$ is a field. Units mod 6: only 1 and 5.
- Mod 6, $2 \times 3 = 0$ though neither factor is 0, so the nonzero elements are not even closed under multiplication.

Worked example: a large power

Find $3^{100} \bmod 7$.

- 7 is prime and does not divide 3, so $3^6 \equiv 1 \pmod{7}$.
- $100 = 6 \times 16 + 4$, so $3^{100} \equiv 3^4 = 81 \pmod{7}$.
- $81 = 77 + 4$, so $3^{100} \equiv 4 \pmod{7}$.

Common errors

- Assuming every group is abelian.
- Using the converse of Lagrange's theorem.
- Thinking every nonzero element of $\mathbb{Z}/n\mathbb{Z}$ is a unit.
- Applying Fermat's little theorem when the modulus is not prime.
- Forgetting that a homomorphism's kernel must be checked before calling it one-to-one.

Key terms

homomorphism: A map between algebraic structures that preserves the operation.

fields: Commutative rings in which every nonzero element has a multiplicative inverse.

Check yourself

- What is the order of 3 in the additive group $\mathbb{Z}/12\mathbb{Z}$?
- How many units does $\mathbb{Z}/12\mathbb{Z}$ have?
- Can a group of order 15 have an element of order 4?

Answers: $12/\gcd(3, 12) = 4$. Four: 1, 5, 7, 11. No: 4 does not divide 15.

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T.1 · Real analysis and topology

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Real analysis and topology

In 1872 Karl Weierstrass presented to the Berlin Academy a function that is continuous everywhere but has a derivative nowhere: its graph is a jagged curve with corners at every scale. Many mathematicians had believed that a continuous function must be smooth except at a few points; Charles Hermite later wrote that he turned away "with fright and horror" from such functions. Weierstrass's example showed that intuition about graphs is not proof. Real analysis rebuilt calculus on exact definitions of limits, continuity and completeness.

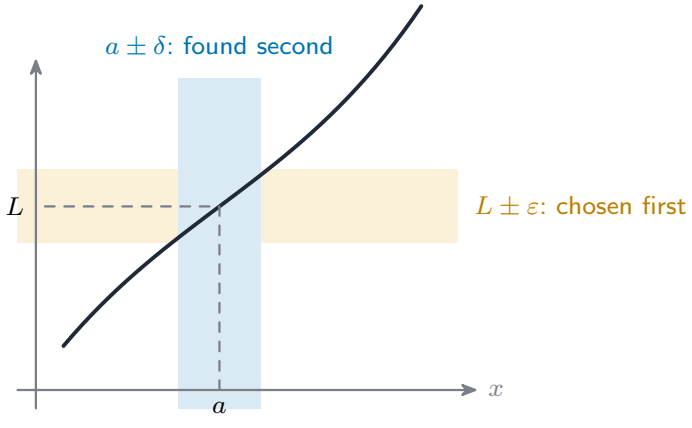
GRE Mathematics analysis questions ask you to:

- Use the epsilon–delta definitions of limits and continuity.
- Work with the **supremum** 上确界 and the completeness of the real numbers.
- Decide whether a set is **compact** 紧致的, connected or complete.
- Use the theorems those properties allow, and know when they fail.

Epsilon and delta

$\lim_{x\to a} f(x) = L$ means: for every $\varepsilon > 0$ there is a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \varepsilon$.

- The order matters: ε is given first, and δ may depend on it.
- Continuity at a is the same statement with $L = f(a)$ and the condition $|x - a| < \delta$.
- Uniform continuity asks for one δ that works for every point at once.



For every epsilon there is a delta: order matters.

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Completeness

- Every nonempty set of reals that is bounded above has a least upper bound, its supremum. The rationals lack this: $\{x \in \mathbb{Q} : x^2 < 2\}$ has no rational supremum.
- Monotone convergence: a bounded monotone sequence converges.
- Bolzano–Weierstrass: every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.
- A Cauchy sequence is one whose terms get arbitrarily close to each other; a space is complete when every Cauchy sequence converges in it.

Compactness

- A set is compact if every open cover has a finite subcover.
- Heine–Borel: in $\mathbb{R}^n$, compact means closed and bounded. This equivalence does not hold in every metric space.
- A continuous function on a compact set is bounded, attains its maximum and minimum, and is uniformly continuous.
- $f(x) = x$ on $(0, 1)$ is bounded but never reaches its supremum 1, because $(0, 1)$ is not compact.

Connectedness

- A set is connected if it cannot be split into two disjoint nonempty open pieces. In $\mathbb{R}$ the connected sets are exactly the intervals.
- Continuous images of connected sets are connected; this is the intermediate value theorem.
- Continuous images of compact sets are compact. Completeness, by contrast, is not preserved by continuous maps.

Worked example: epsilon–delta

Prove $f(x) = x^2$ is continuous at 2.

- Given $\varepsilon > 0$, choose $\delta = \min(1, \varepsilon/5)$.
- If $|x-2| < \delta$, then $1 < x < 3$, so $|x+2| < 5$, and $|x^2-4| = |x-2|\,|x+2| < 5\delta \le \varepsilon$.
- On $[1, 3]$, which is closed and bounded, f attains a minimum: $f(1) = 1$.

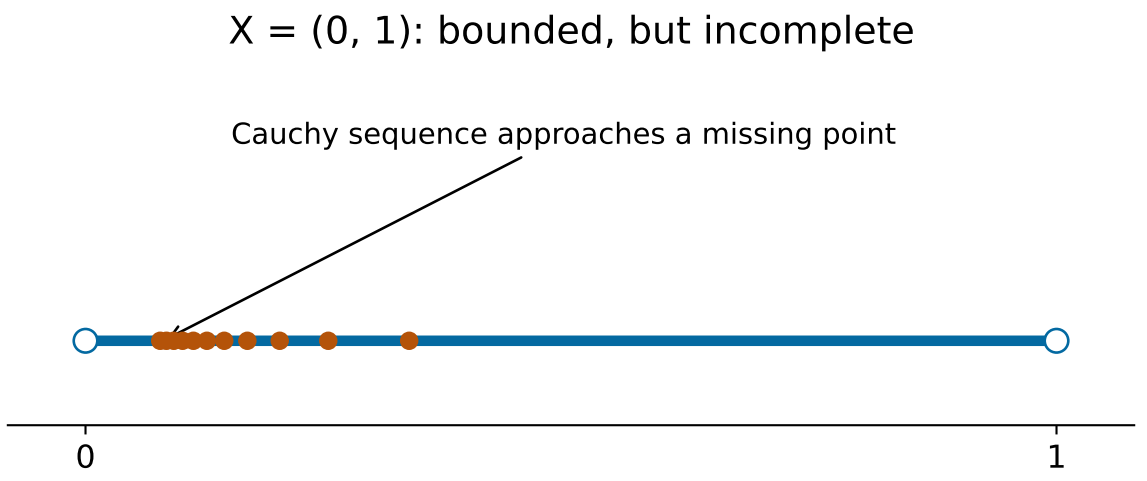
Worked example: an open interval as a space

Take $X = (0, 1)$ with the usual distance.

- X is bounded, and closed in itself (every space is closed in itself).
- The Cauchy sequence $1/(n + 2)$ has no limit in X, so X is not complete, and so not compact.

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- The cover $\{(1/n, 1) : n \geq 2\}$ has no finite subcover. Heine–Borel does not apply, because X is not closed as a subset of $\mathbb{R}$.



(0, 1) is bounded but not complete: a Cauchy sequence can approach a point outside it

Common errors

- Choosing δ before ε, or letting δ depend on x.
- Using "closed and bounded" for compactness outside $\mathbb{R}^n$.
- Assuming a bounded continuous function attains its supremum on any set.
- Confusing complete with compact.
- Thinking the image of a closed set under a continuous map is closed.

Key terms

supremum: The least upper bound of a set.

compact: Having the property that every open cover has a finite subcover.

Check yourself

- What is the supremum of $\{1 - 1/n : n \geq 1\}$? Is it attained?
- Is $[0, \infty)$ compact?
- A continuous function maps $[0, 1]$ onto a set S. Must S be an interval?

Answers: 1, not attained. No: it is not bounded. Yes: a closed bounded interval.

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T.2 · Discrete mathematics, probability and numerical methods

GRE Subject Test · GRE Mathematics

Discrete mathematics, probability and numerical methods

In 1654 a French gambler, the Chevalier de Méré, noticed something odd. Betting on at least one six in four rolls of a die won slightly more often than it lost. But betting on at least one double six in 24 rolls of two dice lost slightly more often, although $4/6 = 24/36$. He asked Blaise Pascal, who wrote to Pierre de Fermat, and their letters founded probability theory. The answer uses the complement: $1-(5/6)^4 = 0.518$, but $1-(35/36)^{24} = 0.491$.

GRE Mathematics discrete and probability questions ask you to:

- Count with permutations, combinations and the complement.
- Set up and solve a **recurrence** 递推关系.
- Use independence, conditional probability and **mutually exclusive** 互斥的 events correctly.
- Use expectation and the binomial distribution.
- Apply simple numerical methods and know their errors.

Counting

- Ordered selections of k from n distinct objects: $n!/(n-k)!$. Unordered: $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$.
- With repetition allowed, strings of length k from n symbols: n^k.
- "At least one" is usually easiest as 1 minus "none".
- Inclusion–exclusion: $|A \cup B| = |A| + |B| - |A \cap B|$.

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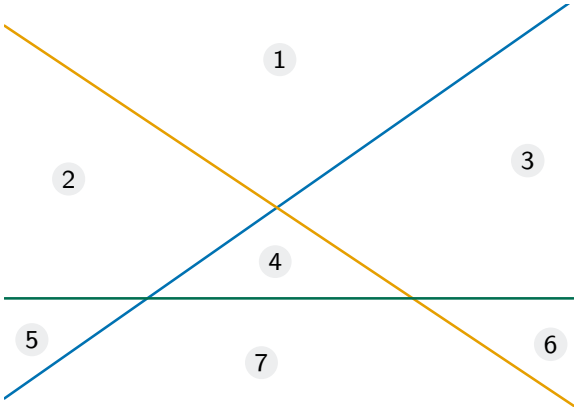
Four labelled independent draws, five equally likely symbols

Event	Count model	Count
ordered strings	5 x 5 x 5 x 5	625
all distinct	5 x 4 x 3 x 2	120
at least one repeat	625 - 120	505

Count "at least one repeat" as the complement of "all distinct"

Recurrences

- A recurrence gives each term from earlier ones. Find the pattern of differences, or solve directly.
- n lines in general position divide the plane into R_n regions with $R_n = R_{n-1} + n$, since the nth line crosses $n-1$ others and so is cut into n pieces, each splitting a region. So $R_n = 1 + \dfrac{n(n+1)}{2}$.
- Linear recurrences like $a_n = a_{n-1} + a_{n-2}$ are solved with a characteristic equation, as for differential equations.



line k crosses $k-1$ lines and adds k regions: $1+1+2+3=7$

parallels or three lines through one point break this

n lines in general position: 1 + n(n+1)/2 regions.

Probability

- Mutually exclusive events cannot both happen: $P(A \cup B) = P(A) + P(B)$.
- Independent events do not affect each other: $P(A \cap B) = P(A)P(B)$. Mutually exclusive events with positive probabilities are never independent.
- Conditional probability: $P(A \mid B) = P(A \cap B)/P(B)$.
- Expectation is linear, $E[X+Y] = E[X]+E[Y]$, even when X and Y are dependent.
- Binomial: $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$, with mean np and variance $np(1-p)$.

Numerical methods

- Bisection halves an interval that contains a sign change; it always converges, slowly.
- Newton's method: $x_{n+1} = x_n - f(x_n)/f'(x_n)$. Near a simple root the number of correct digits roughly doubles each step.
- The trapezoid rule's error shrinks like h^2; Simpson's rule's like h^4. Simpson's rule is exact for cubics.

Worked example: repeated symbols

Four draws, with replacement, from five symbols. Find the probability that at least two agree.

- Ordered strings: $5^4 = 625$. All distinct: $5 \times 4 \times 3 \times 2 = 120$.
- $P(\text{repeat}) = 1 - 120/625 = 505/625 = 101/125$.
- Unordered selections of four distinct symbols number $\binom{5}{4} = 5$; each occurs in $4! = 24$ orders. Divide by 4! only when order does not matter in both the count and the total.

Worked example: Newton's method

Approximate $\sqrt{2}$ as the root of $f(x) = x^2 - 2$, starting from $x_0 = 1$.

- $x_1 = 1 - (1-2)/2 = 1.5$.
- $x_2 = 1.5 - (2.25-2)/3 = 1.41667$.
- $x_3 = 1.414216$, against the true 1.414214: the digits double each step.

Common errors

- Treating mutually exclusive events as independent.
- Adding probabilities of events that can happen together.
- Dividing by $k!$ for an ordered count.
- Forgetting that linearity of expectation needs no independence.
- Using Newton's method where f' is zero or near zero.

Key terms

recurrence: A rule giving each term of a sequence from earlier terms.

mutually exclusive: Describing events that cannot happen together.

Check yourself

- How many regions do 5 lines in general position make?
- A fair coin is tossed 10 times. What is the expected number of heads, and the variance?
- What is the probability of at least one six in three rolls of a die?

Answers: 16. 5 and 2.5. $1 - (5/6)^3 = 91/216$.

T.3 · Complex analysis and residues

GRE Subject Test · GRE Mathematics

Complex analysis and residues

In 1545 Gerolamo Cardano published a formula for solving cubic equations. For some equations with three real roots, such as $x^3 = 15x + 4$, the formula needed the square root of a negative number, $\sqrt{-121}$. Cardano called such quantities "as subtle as they are useless". In 1572 Rafael Bombelli showed that if you calculate with them by the usual rules, they cancel and leave the real root 4. Complex numbers began as a tool for finding real answers, and they still are: residues evaluate real integrals that are hard any other way.

GRE Mathematics complex analysis questions ask you to:

- Use the polar form, de Moivre's theorem and roots of unity.
- Test whether a function is **analytic** 解析的 with the Cauchy–Riemann equations.
- Apply Cauchy's theorem and integral formula.
- Find a **residue** 留数 and use the residue theorem.

Polar form

- $z = re^{i\theta}$ with $r = |z|$. Multiplying multiplies moduli and adds arguments.
- De Moivre: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.
- The nth roots of unity are $e^{2\pi ik/n}$, $k = 0, \dots, n-1$. They sum to 0 for $n \geq 2$.

Analytic functions

Write $f = u + iv$. If f is complex differentiable, then:

$$u_x = v_y, \qquad u_y = -v_x$$

- With continuous partial derivatives, these Cauchy–Riemann equations are also sufficient.
- z^2 passes everywhere; $\overline{z}$ fails everywhere, though it is smooth as a map of the plane.
- An analytic function's real and imaginary parts are harmonic: $u_{xx} + u_{yy} = 0$.

need $u_x = v_y$ and $u_y = -v_x$

$f = z^2$: $u = x^2 - y^2$, $v = 2xy$
$u_x = 2x = v_y$
$u_y = -2y = -v_x$
analytic everywhere

$f = \bar{z}$: $u = x$, $v = -y$
$u_x = 1$, $v_y = -1$
real partials exist, but
nowhere complex differentiable

Cauchy-Riemann: z squared passes, the conjugate fails.

Cauchy's theorems

- Cauchy's theorem: if f is analytic inside and on a closed curve C, then $\oint_C f\,dz = 0$.
- Integral formula: $f(a) = \dfrac{1}{2\pi i}\oint_C \dfrac{f(z)}{z-a}\,dz$ for a inside C.
- Liouville: a bounded function that is analytic on the whole plane is constant.

Residues

Near an isolated singularity a, f has a Laurent series $\sum c_n(z-a)^n$. The residue is c_{-1}.

$$\oint_C f\,dz = 2\pi i \sum \operatorname{Res}(f, a_k) \qquad \text{(anticlockwise, singularities inside } C\text{)}$$

- Simple pole: $\operatorname{Res} = \lim_{z\to a}(z-a)f(z)$.
- Pole of order m: $\operatorname{Res} = \dfrac{1}{(m-1)!}\lim_{z\to a}\dfrac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$.

The residue is the coefficient of $(z-1)^{-1}$

Function	Pole order	Residue at 1
$3/(z-1)$	1	3
$e^z/(z-1)^2$	2	e
$e^{2z}/(z-1)^3$	3	$2e^2$

The residue is the coefficient of $(z-1)^{-1}$ in the Laurent series

Worked example: a double pole

Find $\displaystyle\oint_{|z|=1} \frac{e^z}{z^2}\,dz$.

- $e^z/z^2 = z^{-2} + z^{-1} + \frac{1}{2} + \cdots$, so the residue at 0 is 1.
- The integral is $2\pi i \times 1 = 2\pi i$.
- A double pole does not mean a zero residue; read the coefficient of z^{-1}.

Worked example: a real integral

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

- Close the real line with a large semicircle in the upper half-plane; its contribution goes to zero.
- Inside is the simple pole $z = i$, with residue $\displaystyle\frac{1}{z+i}\Big|_{z=i} = \frac{1}{2i}$.
- The integral is $\displaystyle 2\pi i \times \frac{1}{2i} = \pi$, which agrees with $[\arctan x]_{-\infty}^{\infty}$.

Common errors

- Treating a smooth map of the plane as complex differentiable.
- Reading the residue from the wrong power of $(z-a)$.
- Including poles outside the contour.
- Forgetting the factor $\displaystyle\frac{1}{(m-1)!}$ for higher-order poles.
- Going around the contour clockwise without changing the sign.

Key terms

analytic: Complex differentiable at every point of an open set.

residue: The coefficient of $(z-a)^{-1}$ in the Laurent series of a function at a.

Check yourself

- What is the sum of the three cube roots of unity?
- What is the residue of $\displaystyle\frac{1}{z(z-2)}$ at $z = 2$?
- Can a non-constant function that is analytic on the whole plane be bounded?

Answers: 0. $\frac{1}{2}$. No, by Liouville's theorem.

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C.3 · Differential equations and initial conditions

GRE Subject Test · GRE Mathematics

Differential equations and initial conditions

In 1701 Isaac Newton published, without his name, a short paper on heat. He heated an iron block in a fire, set it to cool in a steady wind, and recorded its temperature against time. He proposed that the rate of cooling is proportional to the difference between the object's temperature and its surroundings. That is a differential equation, $dT/dt = -k(T - T_s)$, and its solution, an exponential decay, is now used to estimate the time of death in forensic science.

GRE Mathematics differential equations questions ask you to:

- Solve separable and first-order linear equations.
- Solve second-order linear equations with constant coefficients from the **characteristic root** 特征根.
- Use an **initial condition** 初始条件 to fix constants.
- Know when a solution exists and is unique.

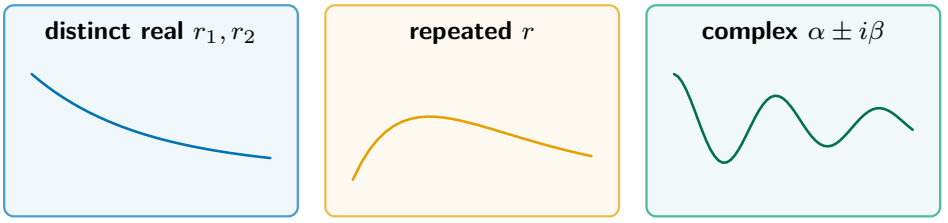
First-order equations

- Separable: $\dfrac{dy}{dx} = g(x)h(y)$. Separate and integrate: $\displaystyle\int \frac{dy}{h(y)} = \int g(x)\,dx$.
- Linear: $y' + p(x)y = q(x)$. Multiply by the integrating factor $\mu = e^{\int p\,dx}$; then $(\mu y)' = \mu q$.
- Growth and decay: $y' = ky$ gives $y = y_0 e^{kt}$.

Second-order with constant coefficients

For $ay'' + by' + cy = 0$, try $y = e^{rt}$, giving $ar^2 + br + c = 0$.

- Two distinct real roots: $y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$.
- A repeated root r: $y = (C_1 + C_2 t)e^{rt}$.
- Complex roots $\alpha \pm i\beta$: $y = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$.
- A second-order equation needs two conditions, such as $y(0)$ and $y'(0)$.



The roots decide the solution form.

Existence and uniqueness

For $y' = f(x, y)$, $y(x_0) = y_0$:

- If f and $\partial f/\partial y$ are continuous near (x_0, y_0), there is exactly one solution near x_0.
- Without this, uniqueness can fail: $y' = 3y^{2/3}$, $y(0) = 0$ has both $y = 0$ and $y = x^3$.
- Solutions can also blow up: $y' = y^2$, $y(0) = 1$ gives $y = 1/(1 - x)$, which exists only for $x < 1$.

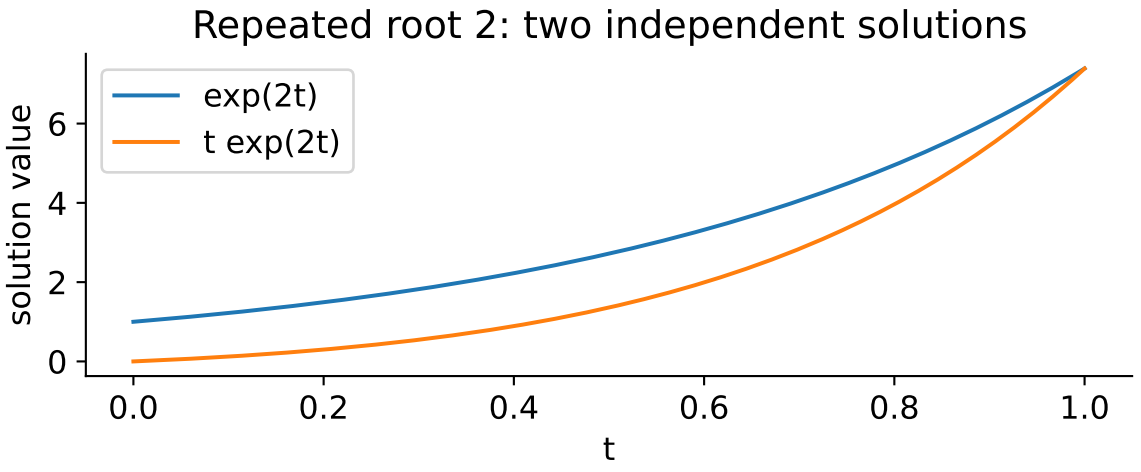
Checking a solution

- Substitute it back into the equation, and check every initial condition.
- Count constants: an nth-order equation's general solution has n of them.

Worked example: a repeated root

Solve $y'' - 4y' + 4y = 0$ with $y(0) = 1$, $y'(0) = 0$.

- $r^2 - 4r + 4 = (r - 2)^2$, a repeated root 2, so $y = (C_1 + C_2 t)e^{2t}$.
- $y(0) = C_1 = 1$; $y'(0) = 2C_1 + C_2 = 0$, so $C_2 = -2$.
- $y = (1 - 2t)e^{2t}$. Check: $y' = -4te^{2t}$, $y'' = (-4 - 8t)e^{2t}$, and $y'' - 4y' + 4y = 0$.



A repeated root 2 gives the two independent solutions e^(2t) and t · e^(2t)

Worked example: cooling

Tea at 90 °C cools in a 20 °C room and is 60 °C after 10 minutes. When does it reach 30 °C?

- $T - 20 = 70e^{-kt}$. At $t = 10$: $40 = 70e^{-10k}$, so $k = \ln(7/4)/10 = 0.0560$ per minute.
- For 30 °C: $10 = 70e^{-kt}$, so $t = \ln 7/k = 1.946/0.0560 = 34.8$ minutes.
- The tea never reaches 20 °C exactly; the difference only decays.

Common errors

- Using two copies of e^{rt} for a repeated root; the second solution is te^{rt}.
- Forgetting the integrating factor's effect on the right-hand side.
- Fixing only one constant of a second-order equation.
- Assuming every initial value problem has a unique solution.
- Dividing by $h(y)$ and losing the constant solution where $h(y) = 0$.

Key terms

characteristic root: A root of the polynomial obtained by substituting e^{rt} into a linear constant-coefficient equation.

initial condition: A value of the solution or its derivative given at a starting point.

Check yourself

- What is the general solution of $y'' + 9y = 0$?
- Solve $y' = 2xy$ with $y(0) = 3$.
- How many constants does the general solution of a third-order linear equation have?

Answers: $C_1 \cos 3t + C_2 \sin 3t$. $y = 3e^{x^2}$. Three.

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A.3 · Groups, cosets and quotient maps

GRE Subject Test · GRE Mathematics

Groups, cosets and quotient maps

In 1974 Ernő Rubik, a design teacher in Budapest, built a cube of 26 small blocks that could twist on every face. Its positions form a group, with about 4.3×10^{19} elements. Take a cube apart and put the pieces back at random, and you get one of twelve times as many arrangements, but only one in twelve can be solved by twisting. The arrangements split into twelve equal families, the **cosets** 陪集 of the cube group, and twisting never moves you from one family to another.

GRE Mathematics group questions ask you to:

- List subgroups and cosets, and use Lagrange's theorem and the index.
- Decide whether a subgroup is a **normal subgroup** 正规子群.
- Build quotient groups and identify them.
- Use the first isomorphism theorem.

Cosets

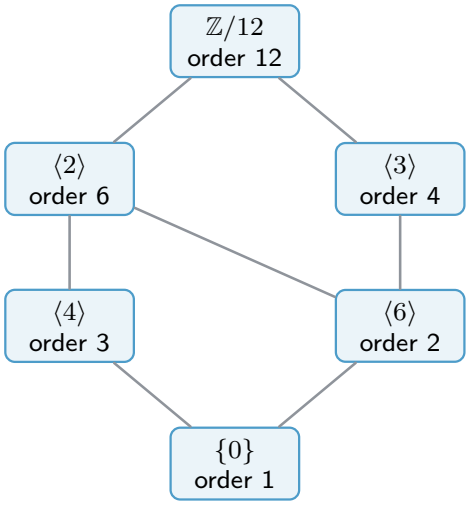
For a subgroup H of G, the left coset of g is $gH = \{gh : h \in H\}$.

- The left cosets partition G into pieces of equal size $|H|$.
- So $|G| = [G : H]\,|H|$, where the index $[G : H]$ is the number of cosets. This is Lagrange's theorem.
- In additive notation the coset is $g + H$.

Cyclic groups

- Every subgroup of a cyclic group is cyclic.
- $\mathbb{Z}/n\mathbb{Z}$ has exactly one subgroup of each order d dividing n, namely $\langle n/d \rangle$.
- $\mathbb{Z}/12\mathbb{Z}$ has subgroups of orders 1, 2, 3, 4, 6 and 12.

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cyclic: exactly one subgroup per divisor; in a general group Lagrange's converse can fail

Subgroups of a cyclic group of order 12.

Normal subgroups and quotients

H is normal when $gH = Hg$ for every g, equivalently $gHg^{-1} = H$.

- Every subgroup of an abelian group is normal. A subgroup of index 2 is always normal.
- If H is normal, the cosets form a group G/H, with $(aH)(bH) = abH$ and order $[G : H]$.
- If H is not normal, this product is not well defined.

The first isomorphism theorem

If $\phi : G \to K$ is a homomorphism, its kernel is normal and

$$G/\ker\phi \cong \operatorname{im}\phi$$

- To identify a quotient G/H, find an onto homomorphism with kernel H.
- Reduction mod 3 maps $\mathbb{Z}/12\mathbb{Z}$ onto $\mathbb{Z}/3\mathbb{Z}$ with kernel $\langle 3 \rangle$, so $(\mathbb{Z}/12\mathbb{Z})/\langle 3 \rangle \cong \mathbb{Z}/3\mathbb{Z}$.

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Reduction modulo 3: three fibres, four elements each

Quotient class	Elements of Z/12Z	Image in Z/3Z
H	0, 3, 6, 9	0
1 + H	1, 4, 7, 10	1
2 + H	2, 5, 8, 11	2

Reduction mod 3 splits Z/12Z into three fibres of four elements each

Worked example: a quotient of Z/18

Let $H = \langle 6 \rangle$ in $\mathbb{Z}/18\mathbb{Z}$.

- $H = \{0, 6, 12\}$, so 6 has order 3.
- The cosets are $r + H = \{r, r+6, r+12\}$ for $r = 0, 1, \ldots, 5$: six cosets of three elements, covering all 18 residues.
- Reduction mod 6 is onto $\mathbb{Z}/6\mathbb{Z}$ with kernel H, so $(\mathbb{Z}/18\mathbb{Z})/H \cong \mathbb{Z}/6\mathbb{Z}$.

Worked example: a subgroup that is not normal

In S_3, let $H = \{e, (12)\}$, and compose permutations right to left.

- $(13)H = \{(13), (13)(12)\} = \{(13), (123)\}$.
- $H(13) = \{(13), (12)(13)\} = \{(13), (132)\}$.
- The left and right cosets differ, so H is not normal. By contrast $A_3 = \{e, (123), (132)\}$ has index 2 and is normal.

Common errors

- Assuming every subgroup is normal.
- Forming a quotient by a subgroup that is not normal.
- Confusing the order of an element with the order of the quotient.
- Composing permutations in the wrong order.
- Thinking a group must have a subgroup for every divisor of its order (only cyclic groups promise this).

Key terms

cosets: The translates gH of a subgroup, which split a group into equal parts.

normal subgroup: A subgroup unchanged by conjugation, $gHg^{-1} = H$, for every group element.

Check yourself

- How many cosets does $\langle 4 \rangle$ have in $\mathbb{Z}/20\mathbb{Z}$?
- A subgroup has index 2. Must it be normal?
- What is $(\mathbb{Z}/12\mathbb{Z})/\langle 4 \rangle$ isomorphic to?

Answers: four. Yes. $\mathbb{Z}/4\mathbb{Z}$.

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A.4 · Rings, ideals and modules

GRE Subject Test · GRE Mathematics

Rings, ideals and modules

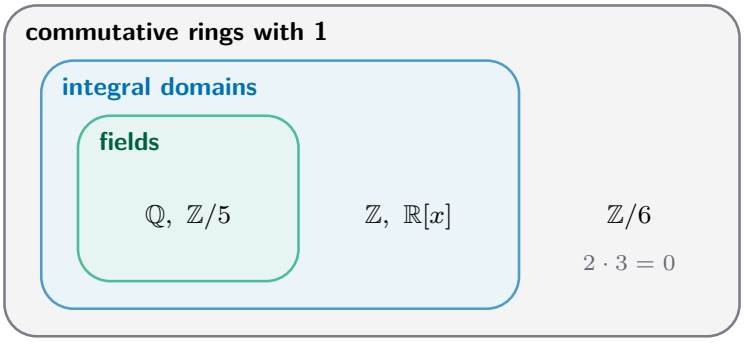
In March 1847 Gabriel Lamé announced to the Paris Academy that he had proved Fermat's Last Theorem. His proof factored numbers of the form $x^n + y^n$ using complex roots of unity, and assumed that factorisation into primes was unique, as it is for ordinary integers. Joseph Liouville pointed out the gap at once. Weeks later a letter arrived from Ernst Kummer: he had shown three years earlier that unique factorisation fails for such numbers. Kummer's "ideal numbers", which Richard Dedekind turned into **ideals** 理想 in 1871, restored a kind of unique factorisation.

GRE Mathematics ring questions ask you to:

- Tell rings, integral domains and fields apart.
- Find units and zero divisors.
- Work with ideals and quotient rings.
- Decide whether an ideal is prime or maximal.
- Recognise **torsion** 挠性 in modules.

Rings, domains and fields

- A ring has addition and multiplication; here rings are commutative with 1.
- An integral domain has no zero divisors: $ab = 0$ forces $a = 0$ or $b = 0$. $\mathbb{Z}$ and $\mathbb{R}[x]$ are domains.
- A field is a ring in which every nonzero element is a unit. Every field is a domain; every finite domain is a field.
- In $\mathbb{Z}/n\mathbb{Z}$, the units are the residues coprime to n, and the other nonzero residues are zero divisors.



Fields, integral domains and rings.

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Ideals and quotients

An ideal I is an additive subgroup with $rI \subseteq I$ for every r in the ring.

- The principal ideal (a) is all multiples of a. Every ideal of $\mathbb{Z}$ is (n).
- The quotient ring R/I has the cosets $r + I$, with the operations of R.
- $\mathbb{Z}/(n) = \mathbb{Z}/n\mathbb{Z}$; $\mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$.

Prime and maximal ideals

- P is prime when R/P is a domain: $ab \in P$ forces $a \in P$ or $b \in P$.
- M is maximal when R/M is a field: no ideal lies strictly between M and R.
- Every maximal ideal is prime. In $\mathbb{Z}$, (p) for a prime p is maximal; (0) is prime but not maximal.
- Unique factorisation fails in $\mathbb{Z}[\sqrt{-5}]$: $6 = 2 \times 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$.

Multiplication in Z/6Z: units and zero divisors

Residue	Product mod 6	Classification
1, 5	1·1 = 1; 5·5 = 1	Units
2, 3, 4	2·3 = 0; 4·3 = 0	Nonzero zero divisors
0	0·a = 0	Zero

In Z/6Z a residue is a unit exactly when it is coprime to 6

Modules

- A module is like a vector space, but its scalars come from a ring.
- An element m is torsion if some nonzero scalar r gives $rm = 0$. In the $\mathbb{Z}$-module $\mathbb{Z}/6\mathbb{Z}$, every element is torsion, since $6m = 0$.
- $\mathbb{Z}$ as a $\mathbb{Z}$-module has no nonzero torsion elements.

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Worked example: Z/12

- Units: residues coprime to 12, namely 1, 5, 7, 11.
- Zero divisors: 2, 3, 4, 6, 8, 9, 10. For example $4 \times 3 = 0$.
- The ideal $(4) = \{0, 4, 8\}$ has quotient $\cong \mathbb{Z}/4\mathbb{Z}$, which has $2 \times 2 = 0$. So (4) is neither prime nor maximal.

Worked example: polynomial quotients

- $x^2 + 1$ has no real root and degree 2, so it is irreducible over $\mathbb{R}$. $\mathbb{R}[x]/(x^2 + 1)$ is a field (it is $\mathbb{C}$), and the ideal is maximal.
- Over $\mathbb{C}$, $x^2 + 1 = (x - i)(x + i)$, so $\mathbb{C}[x]/(x^2 + 1)$ has zero divisors; the ideal is not even prime.
- In $\mathbb{Z}[x]$, the ideal (x) has quotient $\mathbb{Z}$, a domain but not a field: prime, not maximal.

Common errors

- Calling every domain a field.
- Forgetting that a maximal ideal is defined relative to the whole ring.
- Assuming unique factorisation holds in every domain.
- Treating "prime ideal" and "ideal generated by a prime number" as always the same.
- Missing torsion in a finite module.

Key terms

ideals: Additive subgroups of a ring that absorb multiplication by every ring element.

torsion: The property of a module element that some nonzero scalar sends it to zero.

Check yourself

- Is (6) a prime ideal of $\mathbb{Z}$?
- How many units does $\mathbb{Z}/15\mathbb{Z}$ have?
- Is $\mathbb{Q}[x]/(x^2 - 2)$ a field?

Answers: no: $2 \times 3 \in (6)$ but neither factor is. Eight. Yes: $x^2 - 2$ is irreducible over $\mathbb{Q}$.

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A.5 · Polynomials and field extensions

GRE Subject Test · GRE Mathematics

Polynomials and field extensions

On 30 March 1796 Carl Friedrich Gauss, aged 18, found that a regular polygon with 17 sides can be drawn with only a straightedge and compass, something no one since the ancient Greeks had suspected. The reason is that $\cos(2\pi/17)$ lies in a chain of field extensions each of **extension degree** 扩张次数 2. In 1837 Pierre Wantzel used the same idea to prove that two ancient problems, doubling the cube and trisecting an angle, are impossible: they need numbers of degree 3.

GRE Mathematics polynomial questions ask you to:

- Use the division algorithm and the factor theorem.
- Decide whether a polynomial is an **irreducible polynomial** 不可约多项式 over a given field.
- Find extension degrees with the tower law.
- Find minimal polynomials and build finite fields.

Division and roots

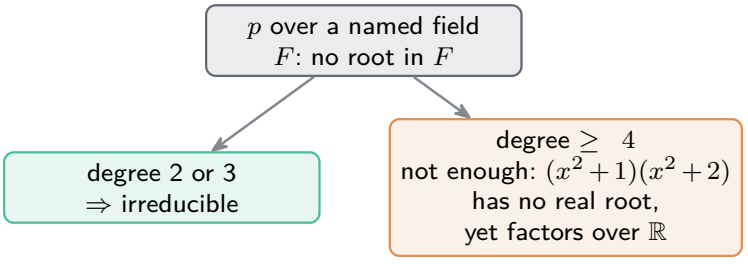
- For polynomials over a field, $f = qg + r$ with $\deg r < \deg g$.
- Factor theorem: a is a root of f exactly when $(x - a)$ divides f.
- A polynomial of degree n over a field has at most n roots.

Irreducibility

Irreducibility depends on the field: $x^2 - 2$ is irreducible over $\mathbb{Q}$ but factors over $\mathbb{R}$.

- Degree 2 or 3: irreducible exactly when there is no root in the field.
- Degree 4 or more: no root is not enough. $(x^2 + 1)(x^2 + 2)$ has no real root but factors.
- Rational root test: a rational root p/q of an integer polynomial has p dividing the constant term and q dividing the leading coefficient.
- Eisenstein: if a prime p divides every coefficient except the leading one, and p^2 does not divide the constant term, the polynomial is irreducible over $\mathbb{Q}$.

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The root test works only up to degree 3.

Extensions and degrees

- $[L : K]$ is the dimension of L as a vector space over K.
- If α has minimal polynomial of degree n over K, then $[K(\alpha) : K] = n$.
- Tower law: $[M : K] = [M : L][L : K]$.
- $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$, since $x^3 - 2$ is irreducible by Eisenstein.

Finite fields

- If f is irreducible of degree n over $\mathbb{F}_p$, then $\mathbb{F}_p[x]/(f)$ is a field with p^n elements.
- There is exactly one field of each prime-power order, up to isomorphism, and none of any other order.
- The nonzero elements of a finite field form a cyclic group.

$F_2[\alpha]$, with $\alpha^2 = \alpha + 1$: all nonzero elements invertible

×	1	α	α + 1
1	1	α	α + 1
α	α	α + 1	1
α + 1	α + 1	1	α

Every nonzero element of $F_2[\alpha]$ has an inverse, so it is a field with four elements

Worked example: a field with four elements

Build $\mathbb{F}_2[x]/(x^2+x+1)$.

- x^2+x+1 is 1 at both $x=0$ and $x=1$, so it has no root in $\mathbb{F}_2$. Having degree 2, it is irreducible, and the quotient is a field.
- Its elements are $0, 1, \alpha, 1+\alpha$, with $\alpha^2 = \alpha + 1$.
- $\alpha(\alpha+1) = \alpha^2 + \alpha = 1$, so $\alpha^{-1} = \alpha + 1$. This field is not $\mathbb{Z}/4\mathbb{Z}$, which has the zero divisor 2.

Worked example: a degree-4 extension

Find $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}]$ and the minimal polynomial of $\sqrt{2}+\sqrt{3}$.

- $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$, and $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$, so the tower law gives $2 \times 2 = 4$.
- Let $x = \sqrt{2}+\sqrt{3}$. Then $x^2 = 5 + 2\sqrt{6}$, so $(x^2-5)^2 = 24$ and $x^4 - 10x^2 + 1 = 0$.
- $\sqrt{2}+\sqrt{3}$ generates the whole degree-4 field, so this quartic is its minimal polynomial.

Common errors

- Calling a polynomial irreducible without naming the field.
- Using the no-root test for degree 4 or more.
- Assuming a field of order 4 is $\mathbb{Z}/4\mathbb{Z}$.
- Multiplying degrees in the tower law without checking each step's degree.
- Applying Eisenstein's criterion with p^2 dividing the constant term.

Key terms

extension degree: The dimension of an extension field as a vector space over its base field.

irreducible polynomial: A polynomial of positive degree that cannot be written as a product of polynomials of lower positive degree over the given field.

Check yourself

- Is $x^3 + x + 1$ irreducible over $\mathbb{F}_2$?
- Is there a field with 6 elements?
- What is $[\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}]$?

Answers: yes: it has no root in $\mathbb{F}_2$ and degree 3. No. 4, by Eisenstein on $x^4 - 2$.

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A.6 · Congruences, divisibility and arithmetic functions

GRE Subject Test · GRE Mathematics

Congruences, divisibility and arithmetic functions

In August 1977 Martin Gardner's column in *Scientific American* printed a message encrypted with a new method, RSA, invented that year by Ron Rivest, Adi Shamir and Leonard Adleman. To read it, you would need to factor a 129-digit number. The authors offered \$100 and estimated that, with the methods of the day, it would take 40 quadrillion years. In 1994 about 600 volunteers sharing their computers over the internet factored it, and the message read: "The magic words are squeamish ossifrage." RSA rests on **congruence** 同余 arithmetic and Euler's **totient** 欧拉函数.

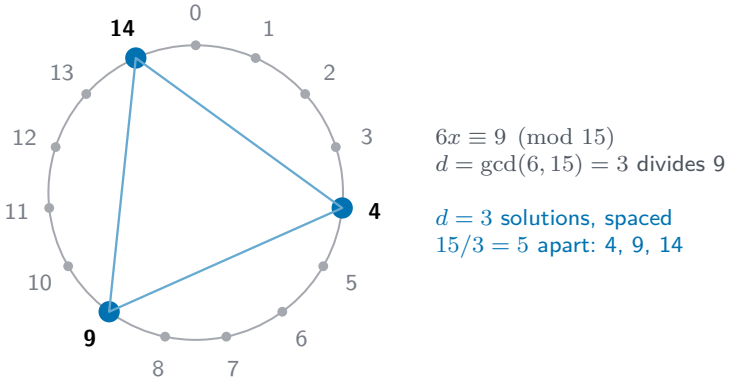
GRE Mathematics number theory questions ask you to:

- Solve linear congruences and find inverses modulo n.
- Use the Chinese remainder theorem.
- Apply Fermat's and Euler's theorems, and Wilson's theorem.
- Compute arithmetic functions such as φ, τ and σ.

Congruences

$a \equiv b \pmod{n}$ means n divides $a - b$.

- $ax \equiv b \pmod{n}$ has a solution exactly when $d = \gcd(a, n)$ divides b. Then it has d solutions modulo n, spaced n/d apart.
- To solve, divide a, b and n by d, solve the smaller congruence, and list the lifts.
- a has an inverse modulo n exactly when $\gcd(a, n) = 1$; find it with the Euclidean algorithm.



$d = gcd(a, n)$ solutions, n/d apart.

The Chinese remainder theorem

If m and n are coprime, the system $x \equiv a \pmod{m}$, $x \equiv b \pmod{n}$ has exactly one solution modulo mn.

- Equivalently, $\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.
- If m and n share a factor, a solution exists only when $a \equiv b$ modulo $\gcd(m, n)$.

Fermat, Euler and Wilson

- Euler: if $\gcd(a, n) = 1$, then $a^{\varphi(n)} \equiv 1 \pmod{n}$. For a prime p this is Fermat's little theorem, $a^{p-1} \equiv 1$.
- Wilson: p is prime exactly when $(p-1)! \equiv -1 \pmod{p}$.
- These theorems need their conditions: Euler's needs a to be a unit.

Arithmetic functions

If $n = p_1^{a_1} \cdots p_k^{a_k}$:

$$\varphi(n) = n \prod \left(1 - \frac{1}{p_i}\right), \qquad \tau(n) = \prod (a_i + 1), \qquad \sigma(n) = \prod \frac{p_i^{a_i+1} - 1}{p_i - 1}$$

- φ counts residues coprime to n; τ counts divisors; σ adds them.
- All three are multiplicative: $f(mn) = f(m)f(n)$ when $\gcd(m, n) = 1$.

Worked example: a congruence with several solutions

Solve $8x \equiv 12 \pmod{20}$.

- $\gcd(8, 20) = 4$, which divides 12, so there are 4 solutions.
- Divide by 4: $2x \equiv 3 \pmod{5}$, so $x \equiv 4 \pmod{5}$ (since $2 \times 4 = 8 \equiv 3$).
- Modulo 20: $x = 4, 9, 14, 19$. Check: $8 \times 9 = 72 = 60 + 12$.

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$6x \equiv 9 \pmod{15}$: lift the reduced solution $x \equiv 4 \pmod{5}$

x modulo 15	6x modulo 15	Valid solution?
4	9	Yes
9	9	Yes
14	9	Yes

Solve $2x \equiv 3 \pmod{5}$ to get $x \equiv 4$, then lift to three solutions modulo 15

Worked example: inverses and Euler

- The inverse of 7 modulo 20 is 3, since $21 \equiv 1$.
- $\varphi(20) = 20 \times \frac{1}{2} \times \frac{4}{5} = 8$, so $7^8 \equiv 1 \pmod{20}$.
- But $2^8 = 256 \equiv 16 \pmod{20}$, not 1: Euler's theorem does not apply, because $\gcd(2, 20) = 2$.
- CRT: $x \equiv 2 \pmod{3}$ and $x \equiv 3 \pmod{5}$ give $x \equiv 8 \pmod{15}$.

Common errors

- Dividing a congruence by a number that shares a factor with the modulus, without dividing the modulus.
- Giving one solution when there are $\gcd(a, n)$ of them.
- Using Euler's theorem for a non-unit.
- Using the Chinese remainder theorem with moduli that are not coprime.
- Forgetting that φ is multiplicative only for coprime factors.

Key terms

congruence: The relation $a \equiv b \pmod{n}$, meaning n divides $a - b$.

totient: Euler's function $\varphi(n)$, the number of residues from 1 to n that are coprime to n.

Check yourself

- What is $\varphi(36)$?
- What is $6!$ mod 7?
- How many positive divisors does 72 have?

Answers: $36\times\frac{1}{2}\times\frac{2}{3}=12$. 6, that is -1, as Wilson's theorem says. $(3+1)(2+1)=12$.

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T.4 · Sequences, series and uniform convergence

GRE Subject Test · GRE Mathematics

Sequences, series and uniform convergence

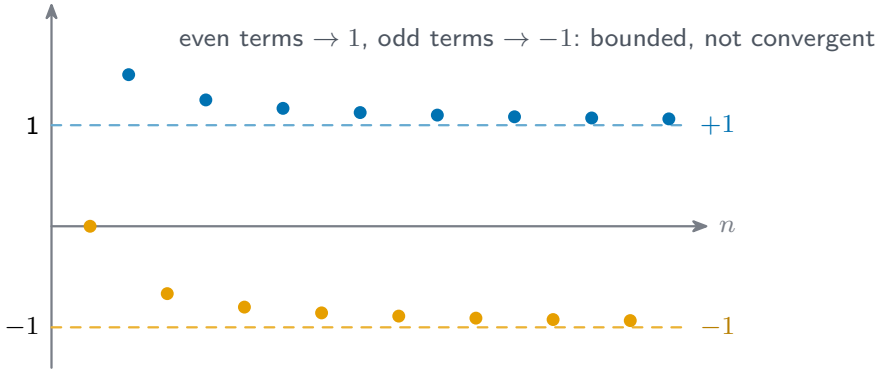
In his famous 1821 textbook, Augustin-Louis Cauchy stated that a convergent series of continuous functions has a continuous sum. In 1826 the young Norwegian Niels Henrik Abel pointed out, politely, that "this theorem admits exceptions". The Fourier series $\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots$ converges everywhere, every term is continuous, yet its sum jumps at $x = \pi$. The missing idea, found decades later, was **uniform convergence** 一致收敛: the series must converge at the same rate everywhere.

GRE Mathematics series questions ask you to:

- Decide whether a sequence converges, using monotonicity and the **Cauchy sequence** 柯西序列 idea.
- Choose and apply a convergence test for a series.
- Separate absolute from conditional convergence.
- Tell pointwise from uniform convergence, and know what uniform convergence preserves.

Sequences

- A convergent sequence is bounded; a bounded sequence need not converge: $(-1)^n + 1/n$ has subsequences tending to 1 and -1.
- A bounded monotone sequence converges.
- In $\mathbb{R}$, a sequence converges exactly when it is a Cauchy sequence.



Bounded, not convergent: two subsequential limits.

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Tests for series

- nth-term test: if $a_n \not\to 0$, the series diverges. If $a_n \to 0$, nothing follows ($\sum 1/n$ diverges).
- Comparison: compare with $\sum 1/n^p$ (converges for $p > 1$) or a geometric series.
- Ratio test: if $|a_{n+1}/a_n| \to L$, the series converges absolutely for $L < 1$ and diverges for $L > 1$. $L = 1$ decides nothing.
- Integral test for positive decreasing terms; alternating series test for alternating terms that decrease to 0.

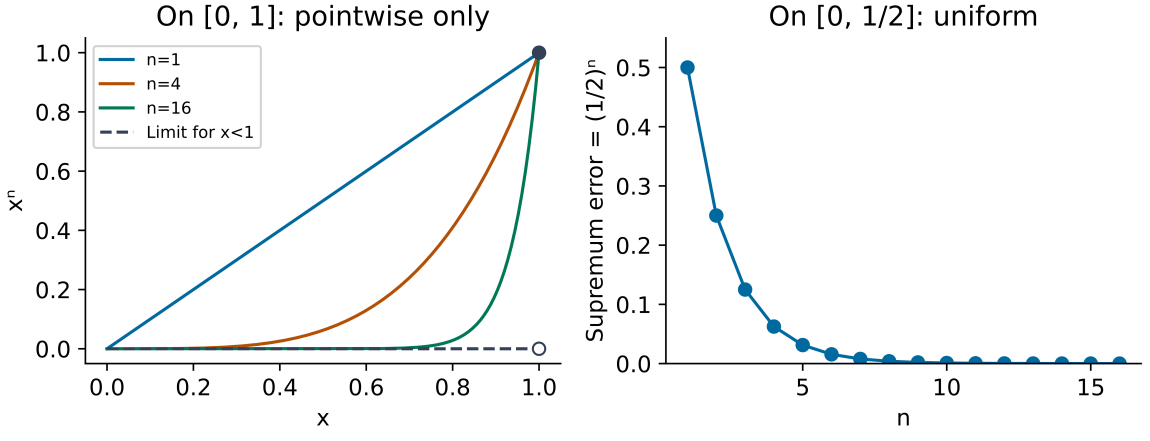
Absolute and conditional

- $\sum a_n$ converges absolutely when $\sum |a_n|$ converges; absolute convergence implies convergence.
- $1 - \frac{1}{2} + \frac{1}{3} - \cdots$ converges (to $\ln 2$) but not absolutely: conditional convergence.
- Riemann: a conditionally convergent series can be rearranged to sum to any real number.

Uniform convergence

$f_n \to f$ uniformly on D when $\sup_D |f_n - f| \to 0$: one index works for every point.

- A uniform limit of continuous functions is continuous. So if the limit is discontinuous, the convergence was not uniform.
- Uniform convergence on a bounded interval allows integrating term by term.
- Weierstrass M-test: if $|f_n(x)| \le M_n$ for all x and $\sum M_n$ converges, then $\sum f_n$ converges uniformly.



x^n converges only pointwise on [0, 1], but uniformly on [0, 1/2]

Worked example: powers of x

Study $f_n(x) = x^n$ on $[0,1]$ and on $[0,a]$ with $0 < a < 1$.

- On $[0,1]$ the limit is 0 for $x < 1$ and 1 at $x = 1$. The limit is discontinuous, so the convergence is not uniform; indeed $\sup_{[0,1)} x^n = 1$.
- On $[0,a]$, $\sup x^n = a^n \to 0$: uniform convergence to 0.
- The domain, not just the formula, decides uniformity.

Worked example: choosing tests

- $\sum \dfrac{n!}{n^n}$: the ratio is $\left(\dfrac{n}{n+1}\right)^n \to \dfrac{1}{e} < 1$, so it converges.
- $\sum \dfrac{\sin nx}{n^2}$: $|\sin nx/n^2| \le 1/n^2$, so by the M-test it converges uniformly on $\mathbb{R}$, and its sum is continuous.
- $\sum \dfrac{(-1)^n}{\sqrt{n}}$: converges by the alternating test, but $\sum 1/\sqrt{n}$ diverges, so the convergence is conditional.

Common errors

- Concluding convergence from $a_n \to 0$.
- Treating ratio limit 1 as a decision.
- Assuming that pointwise convergence preserves continuity or integrals.
- Rearranging a conditionally convergent series and keeping the sum.
- Forgetting that uniform convergence depends on the domain.

Key terms

Cauchy sequence: A sequence whose terms eventually stay arbitrarily close to one another.

uniform convergence: Convergence in which one index makes the error small at every point of the domain at once.

Check yourself

- Does $\sum \dfrac{n}{2^n}$ converge, and to what?
- Does $f_n(x) = x/n$ converge uniformly on $\mathbb{R}$?
- Does $\sum \dfrac{1}{n \ln n}$ (from $n = 2$) converge?

Answers: yes, to 2. No: $\sup |x/n|$ is infinite, though it converges uniformly on any bounded interval. No, by the integral test.

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T.5 · Open sets, compactness and connectedness

GRE Subject Test · GRE Mathematics

Open sets, compactness and connectedness

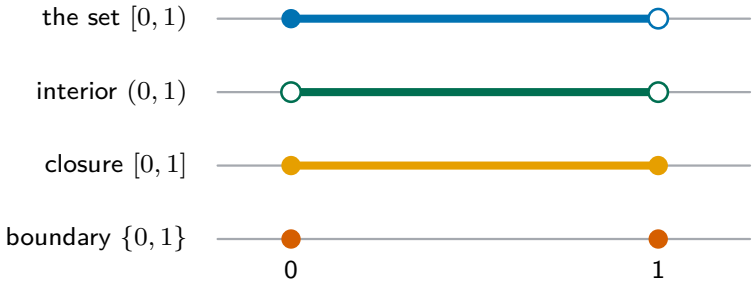
In 1883 Georg Cantor described a set built by a simple rule. Start with $[0,1]$, remove the open middle third, then remove the middle third of each remaining piece, and repeat for ever. The lengths removed add to $\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\cdots=1$, so the Cantor set has total length zero. Yet it has as many points as the whole interval, it is closed and bounded, and it contains no interval at all. Examples like this showed that sets need careful definitions: open, closed, compact, connected.

GRE Mathematics topology questions ask you to:

- Find the interior, closure and boundary of a set.
- Use the **relative topology** 相对拓扑 of a subspace.
- Decide whether a set is compact.
- Decide **connectedness** 连通性 and path-connectedness.

Open and closed

- A set is open if every point has a small ball around it inside the set. It is closed if its complement is open, equivalently if it contains all its limit points.
- Open and closed are not opposites: $\mathbb{R}$ and $\varnothing$ are both; $[0,1)$ is neither; $\mathbb{Q}$ is neither.
- Interior: the largest open subset. Closure: the smallest closed superset. Boundary: closure minus interior.

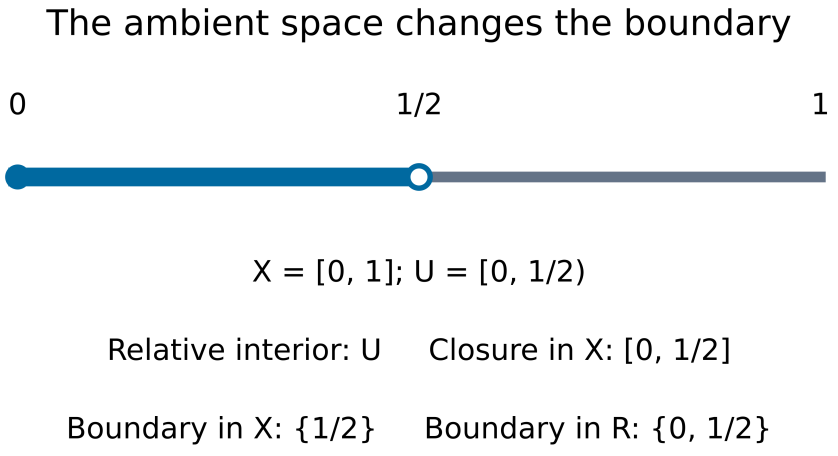


Interior, closure and boundary of [0, 1).

Subspaces

In a subspace X, the open sets are the intersections of X with open sets of the larger space.

- $[0, \frac{1}{2})$ is open in $X = [0, 1]$, because it equals $X \cap (-1, \frac{1}{2})$, though it is not open in $\mathbb{R}$.
- Interior, closure and boundary must be computed in the stated space.



Interior, closure and boundary depend on the space: U's boundary is {1/2} in X but {0, 1/2} in $\mathbb{R}$

Compactness

- Compact: every open cover has a finite subcover. In a metric space, equivalently every sequence has a subsequence converging in the set.
- In $\mathbb{R}^n$: compact exactly when closed and bounded. The Cantor set is compact.
- A closed subset of a compact set is compact; a compact subset of a metric space is closed.
- Continuous images of compact sets are compact.

Connectedness

- Connected: the set cannot be split into two disjoint nonempty relatively open parts. In $\mathbb{R}$ the connected sets are the intervals.
- Path-connected (any two points joined by a continuous path) implies connected. The converse fails: the topologist's sine curve, the graph of $\sin(1/x)$ for $x > 0$ together with the segment $\{0\} \times [-1, 1]$, is connected but not path-connected.

- Continuous images of connected sets are connected. The Cantor set is totally disconnected: its only connected pieces are single points.

Worked example: a relative set

In $X = [0, 2]$, let $A = [0, 1)$.

- $A = X \cap (-1, 1)$, so A is open in X, and its interior in X is A itself. The point 0 is interior because a small ball in X around 0 has no negative part.
- Its closure is $[0, 1]$ and its boundary in X is $\{1\}$. A is not closed in X: $1 - 1/n \to 1 \notin A$.
- In $\mathbb{R}$ the interior and boundary would be $(0, 1)$ and $\{0, 1\}$.

Worked example: the Cantor set

- The Cantor set is the set of numbers in $[0, 1]$ with a base-3 expansion using only the digits 0 and 2.
- $\frac{1}{4} = 0.020202\ldots_3$ is in the set, though it is not an endpoint of any removed interval.
- It is closed (an intersection of closed sets), bounded, so compact; uncountable; of length zero; and contains no interval.

Common errors

- Treating "not open" as "closed".
- Computing interior or boundary in $\mathbb{R}$ when the question names a subspace.
- Using "closed and bounded" for compactness outside $\mathbb{R}^n$.
- Assuming connected sets are path-connected.
- Thinking a set of length zero must be countable.

Key terms

relative topology: The open sets of a subspace, formed by intersecting it with open sets of the larger space.

connectedness: The property of a space that cannot be split into two disjoint nonempty open parts.

Check yourself

- Is $\mathbb{Q}$ open in $\mathbb{R}$? Closed?
- Is $[0, 1] \cup [2, 3]$ connected?
- A continuous function maps $[0, 1]$ into $\mathbb{R}$. What kind of set is the image?

Answers: neither. No. A closed bounded interval (possibly a single point).

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C.4 · Taylor expansions and power-series endpoints

GRE Subject Test · GRE Mathematics

Taylor expansions and power-series endpoints

Around 1400, in Kerala in southern India, the astronomer Madhava of Sangamagrama found an infinite series for π:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

along with series for sine and cosine, more than two centuries before James Gregory and Gottfried Leibniz found them in Europe. His followers recorded the results in verse. Madhava also gave correction terms to speed up the very slow convergence. The series sits exactly at the edge of the interval where the arctangent series converges, a point where careful testing is needed.

GRE Mathematics power-series questions ask you to:

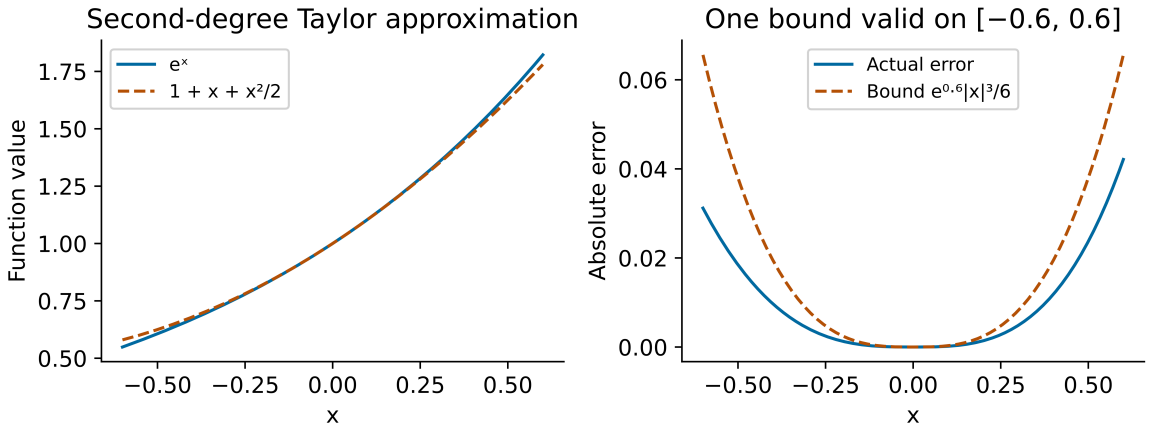
- Find Taylor polynomials and bound the **remainder** 余项.
- Find the **radius of convergence** 收敛半径 and test the endpoints.
- Differentiate and integrate power series term by term.
- Use standard series to find sums and coefficients.

Taylor polynomials

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

- Lagrange remainder: $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some c between a and x.
- To bound the error, bound $|f^{(n+1)}|$ on the whole interval.

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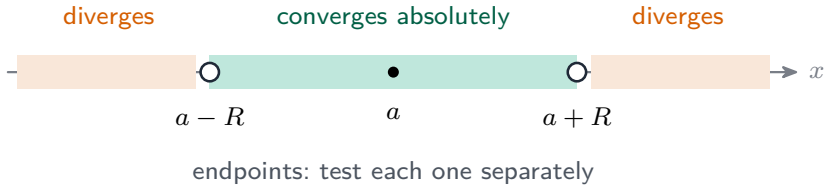


One Lagrange bound covers the whole interval $[-0.6,\ 0.6]$

Radius of convergence

A power series $\sum c_n(x-a)^n$ converges absolutely for $|x-a| < R$ and diverges for $|x-a| > R$.

- Find R with the ratio test: $R = \lim |c_n/c_{n+1}|$ when this limit exists.
- At $|x-a| = R$ anything can happen. Test each endpoint separately.
- $\sum n!\,x^n$ has $R = 0$; $\sum x^n/n!$ has $R = \infty$.



Inside R, outside R, and the endpoints.

Term-by-term operations

- Inside the radius, a power series can be differentiated and integrated term by term. The radius stays the same.
- The endpoints can change: differentiating can lose an endpoint, and integrating can gain one.

Standard series

- $e^x = \sum x^n/n!$; $\sin x = x - x^3/3! + \cdots$; $\cos x = 1 - x^2/2! + \cdots$, all for every x.
- $\dfrac{1}{1-x} = \sum x^n$ for $|x| < 1$.
- $\ln(1+x) = x - x^2/2 + x^3/3 - \cdots$ for $-1 < x \le 1$.
- $\arctan x = x - x^3/3 + x^5/5 - \cdots$ for $-1 \le x \le 1$.

Worked example: endpoints

Find where $\sum_{n\ge1} x^n/n$ converges, and where its derivative does.

- The ratio test gives $R = 1$.
- At $x = 1$: the harmonic series diverges. At $x = -1$: the alternating harmonic series converges. Interval: $[-1, 1)$.
- The derivative is $\sum x^{n-1} = 1/(1-x)$, whose terms do not tend to 0 at ±1. Interval: $(-1, 1)$. The derivative lost the endpoint -1.

Worked example: an error bound

Approximate $e^{0.1}$ with P_2.

- $P_2(0.1) = 1 + 0.1 + 0.005 = 1.105$.
- On $[0, 0.1]$, $f''' = e^x < 2$, so $|R_2| \le 2(0.1)^3/6 = 1/3000 \approx 0.00033$.
- The true value is 1.10517, an error of 0.00017, within the bound. For Madhava's series, the alternating test bounds the error after n terms by the next term, $1/(2n+1)$: a thousand terms give only about three digits of $\pi/4$.

Common errors

- Deciding the endpoints with the ratio test, which gives 1 there.
- Assuming the derivative series has the same interval, endpoints included.
- Bounding the remainder at one point instead of over the interval.
- Forgetting the factorial in Taylor coefficients.
- Using $\ln(1+x)$ at $x = -1$.

Key terms

remainder: The difference between a function and its Taylor polynomial.

radius of convergence: The distance from the centre within which a power series converges absolutely.

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Check yourself

- What is the coefficient of x^4 in the Maclaurin series of $\cos x$?
- What is the interval of convergence of $\sum x^n/n^2$?
- What is the radius of convergence of $\sum 2^n x^n$?

Answers: $\frac{1}{24}$. $[-1, 1]$. $\frac{1}{2}$.

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C.5 · Improper integrals and geometric applications

GRE Subject Test · GRE Mathematics

Improper integrals and geometric applications

In 1641 Evangelista Torricelli, Galileo's student, rotated the curve $y = 1/x$, for $x \geq 1$, around the x-axis. The solid stretches to infinity, yet he proved that its volume is finite: exactly π. Its surface area, however, is infinite. A horn that could be filled with a finite amount of paint could never be painted. The shape, later called Gabriel's horn, puzzled philosophers for decades. It shows how an **improper integral** 反常积分 can converge or diverge depending on what is being integrated.

GRE Mathematics questions on improper integrals and geometry ask you to:

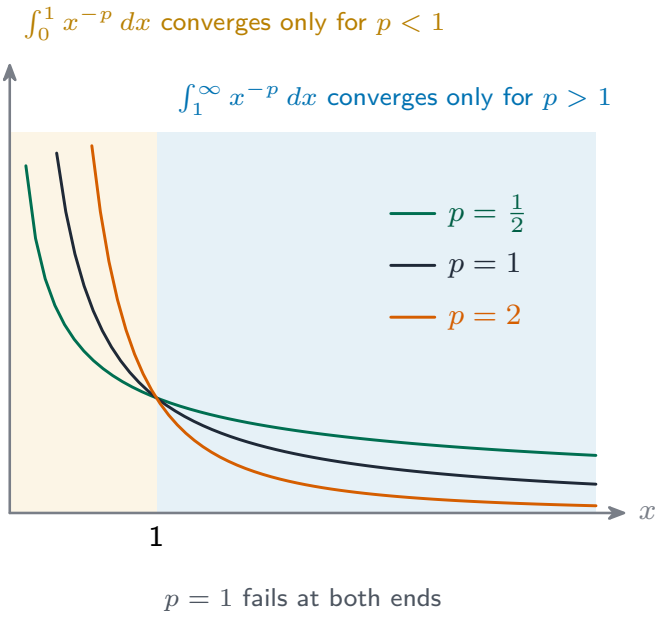
- Decide whether improper integrals converge, using p-tests and comparison.
- Separate an ordinary improper integral from a **principal value** 主值.
- Find areas, volumes by discs and shells, arc lengths and surface areas.

Improper integrals

An integral with an infinite limit or an infinite integrand is defined as a limit.

- $\int_1^\infty x^{-p}\,dx$ converges exactly when $p > 1$.
- $\int_0^1 x^{-p}\,dx$ converges exactly when $p < 1$.
- If the integral is improper at both ends, split it: both parts must converge separately.
- Comparison: if $0 \leq f \leq g$ and $\int g$ converges, so does $\int f$. Near infinity, $e^{-x^2} \leq e^{-x}$ for $x \geq 1$.

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Which p-integrals converge, and where.

Principal values

- $\displaystyle\int_{-1}^{1} \frac{dx}{x}$ diverges: each half diverges on its own.
- With symmetric cut-offs, $\displaystyle\lim_{\varepsilon\to 0}\left(\int_{-1}^{-\varepsilon} + \int_{\varepsilon}^{1}\right)\frac{dx}{x} = 0$. This is the principal value, not the value of the ordinary integral.

Geometry with integrals

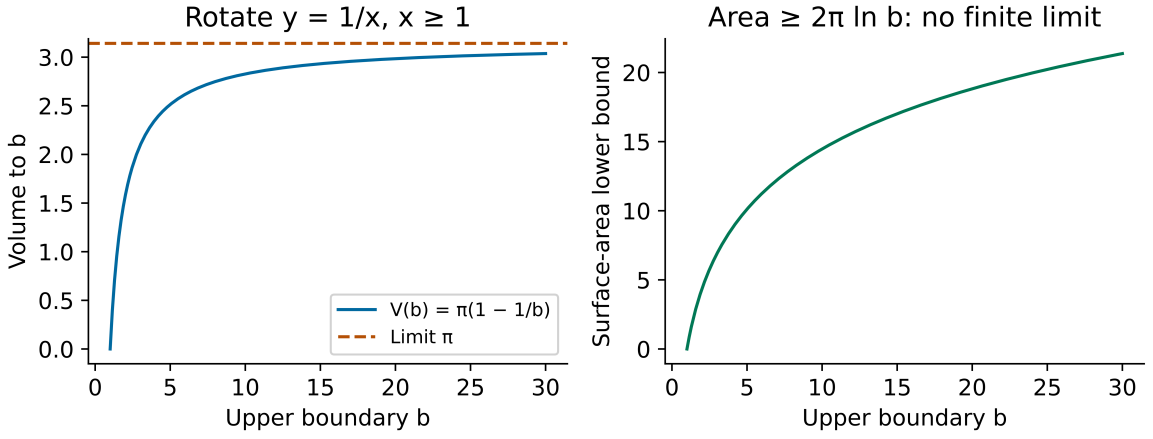
- Area between curves: $\int (\text{top} - \text{bottom})\,dx$.
- Volume by discs, rotating about the x-axis: $\pi \int f(x)^2\,dx$.
- Volume by shells, rotating about the y-axis: $2\pi \int x\, f(x)\,dx$.
- Arc length: $\int \sqrt{1 + f'(x)^2}\,dx$. Surface area about the x-axis: $2\pi \int f\sqrt{1 + f'^2}\,dx$.

Worked example: Gabriel's horn

Rotate $y = 1/x$, $x \geq 1$, about the x-axis.

- Volume: $\displaystyle \pi \int_1^b \frac{dx}{x^2} = \pi\left(1 - \frac{1}{b}\right) \to \pi$.

- Surface area: $2\pi \int_1^b \frac{1}{x}\sqrt{1+x^{-4}}\,dx \geq 2\pi \int_1^b \frac{dx}{x} = 2\pi \ln b \to \infty$.
- $\int_0^\infty x^{-p}\,dx$ diverges for every p: near 0 it needs $p < 1$, near infinity $p > 1$.



Gabriel's horn: finite volume π, infinite surface area

Worked example: shells and length

- Rotate the region under $y = x^2$, $0 \leq x \leq 1$, about the y-axis: $V = 2\pi \int_0^1 x \cdot x^2\,dx = \frac{\pi}{2}$.
- Arc length of $y = x^{3/2}$ from 0 to 4: $\int_0^4 \sqrt{1+\frac{9}{4}x}\,dx = \frac{8}{27}\left[\left(1+\frac{9}{4}x\right)^{3/2}\right]_0^4 = \frac{8}{27}\left(10^{3/2}-1\right) = 9.07$.
- $\int_0^\infty x^n e^{-x}\,dx = n!$, so $\int_0^\infty x^3 e^{-x}\,dx = 6$.

Common errors

- Evaluating an integral across a singularity as if it were ordinary.
- Reporting a principal value as the integral's value.
- Using $\pi \int f^2$ when rotating about the other axis.
- Forgetting to split an integral that is improper at both ends.
- Thinking a finite volume implies a finite surface area.

Key terms

improper integral: An integral defined as a limit, because a limit of integration or the integrand is infinite.

principal value: A limit with symmetric cut-offs around a singularity, which can exist when the ordinary improper integral diverges.

Check yourself

- Does $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$ converge, and to what?
- Does $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converge?
- Rotate $y=\sqrt{x}$, $0\le x\le 4$, about the x-axis. What is the volume?

Answers: yes, to 2. Yes, by comparison with $1/x^2$. $\pi\int_0^4 x\,dx = 8\pi$.

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C.6 · Multivariable extrema and constrained optimisation

GRE Subject Test · GRE Mathematics

Multivariable extrema and constrained optimisation

In 1613 Johannes Kepler bought wine for his wedding and watched the merchant measure each barrel's volume with a single rod pushed diagonally through the bung hole. Suspicious, Kepler worked out the mathematics and published it in 1615. He found that the Austrian barrel's proportions came very close to giving the largest volume for a given diagonal. Near a maximum the volume hardly changes with the shape, so the rough rule worked well. A flat top is the sign of a critical point.

GRE Mathematics optimisation questions ask you to:

- Find critical points of functions of several variables.
- Classify them with the **Hessian** 海森矩阵 and know when the test fails.
- Find global extrema on closed bounded regions.
- Use a **Lagrange multiplier** 拉格朗日乘子 for constrained problems.

Critical points

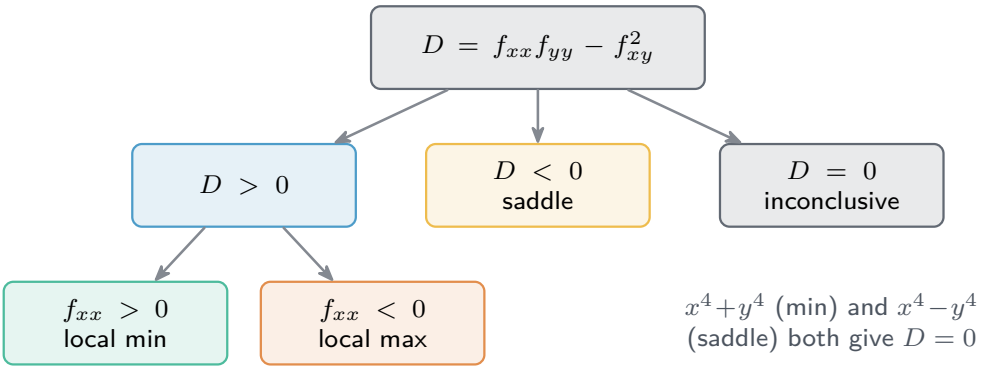
A critical point of $f(x, y)$ is where $\nabla f = 0$, or where it does not exist.

- Local extrema of a differentiable function can occur only at critical points.
- Not every critical point is an extremum: $x^2 - y^2$ has a saddle at the origin.

The second-derivative test

With $D = f_{xx} f_{yy} - f_{xy}^2$, the determinant of the Hessian, at a critical point:

- $D > 0$ and $f_{xx} > 0$: local minimum. $D > 0$ and $f_{xx} < 0$: local maximum.
- $D < 0$: saddle point.
- $D = 0$: the test says nothing. $x^4 + y^4$ (a minimum) and $x^4 - y^4$ (a saddle) both have $D = 0$ at the origin.



The second-derivative test, including its blind spot.

Global extrema

A continuous function on a closed bounded region attains a global maximum and minimum.

- Check the critical points inside the region.
- Check the boundary, by parametrising it or with Lagrange multipliers.
- Compare all the candidate values.

Lagrange multipliers

To find extrema of f subject to $g = c$, solve

$$\nabla f = \lambda \nabla g, \qquad g = c$$

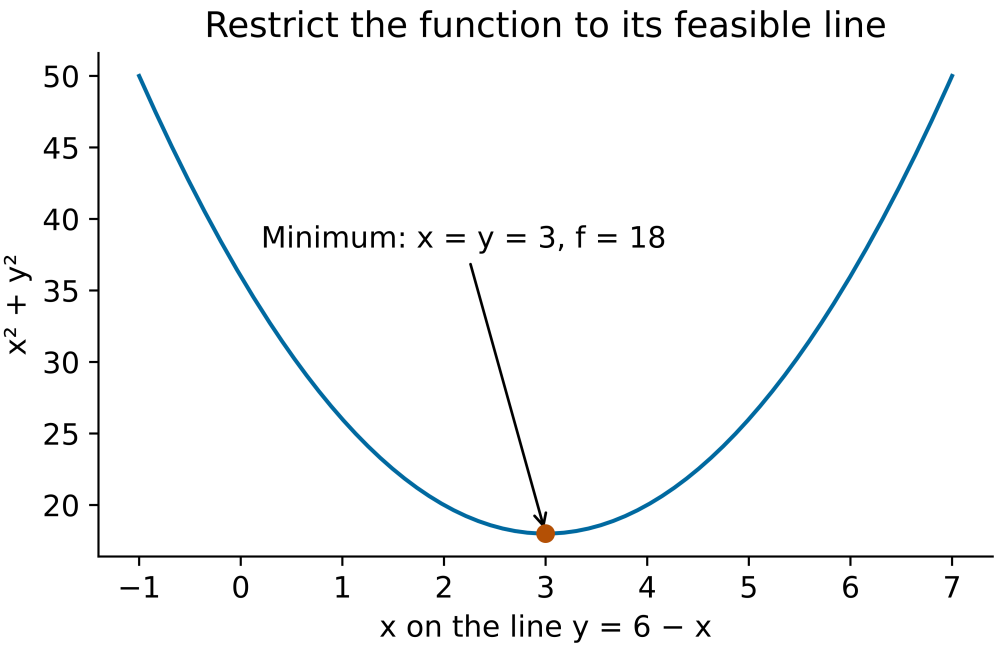
- This works where $\nabla g \neq 0$; points where $\nabla g = 0$ must be checked separately.
- The method gives candidates; compare their values to decide which is the maximum or minimum.
- With two constraints, use $\nabla f = \lambda \nabla g + \mu \nabla h$.

Worked example: classifying

- $f = x^2 - y^2$: the Hessian is $\operatorname{diag}(2, -2)$, $D = -4 < 0$, so the origin is a saddle.
- $g = x^4 + y^4$: the Hessian at 0 is zero, so the test is inconclusive. But $g \geq 0$ with equality only at the origin, so it is a strict global minimum.
- $h = x^2 + xy + y^2$: $D = 2 \times 2 - 1 = 3 > 0$ and $h_{xx} = 2 > 0$, a minimum.

Worked example: constraints

- Minimise $x^2 + y^2$ on $x + y = 6$: $(2x, 2y) = \lambda(1, 1)$ gives $x = y = 3$, value 18. Substituting $y = 6 - x$ gives the same result.
- Maximise xy on $x^2 + y^2 = 8$: $(y, x) = \lambda(2x, 2y)$ gives $y^2 = x^2$. With $x = y = 2$, $xy = 4$; with $x = -y$, $xy = -4$. So the maximum is 4 and the minimum -4.
- A box with no lid and volume 32 has least surface area when it is $4 \times 4 \times 2$.



Substitute the constraint first: the minimum is at $x = y = 3$, where $x^2 + y^2 = 18$

Common errors

- Treating $D = 0$ as a saddle or a minimum.
- Forgetting the boundary when finding global extrema.
- Reading every Lagrange candidate as a maximum.
- Ignoring points where $\nabla g = 0$.
- Using f_{xx} alone to classify a point.

Key terms

Hessian: The matrix of second partial derivatives, used to study the shape of a function near a point.

Lagrange multiplier: The scalar λ in $\nabla f = \lambda \nabla g$ at a constrained extremum.

Check yourself

- Classify the origin for $f = xy$.
- What is the maximum of $x + y$ on the unit circle?
- At a critical point, $f_{xx} = 1$, $f_{yy} = 4$, $f_{xy} = 2$. What does the test say?

Answers: a saddle ($D = -1$). $\sqrt{2}$, at $x = y = 1/\sqrt{2}$. Nothing: $D = 0$.

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C.7 · Coordinate changes and vector integral theorems

GRE Subject Test · GRE Mathematics

Coordinate changes and vector integral theorems

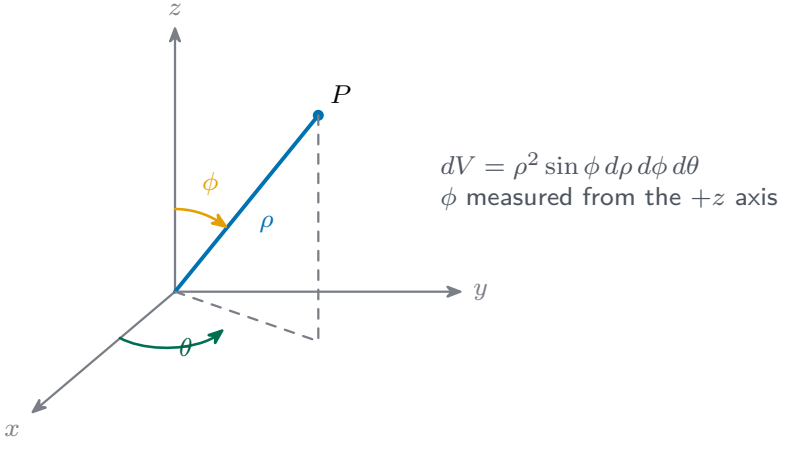
In 1828 George Green, a self-taught miller from Nottingham with about a year of school, paid for the printing of *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*. Only about 51 people subscribed, mostly friends. The essay contained what we now call Green's theorem. It was almost forgotten until 1845, when the young William Thomson (later Lord Kelvin) found a copy and spread it across Europe. Green's idea, that an integral over a region equals an integral around its edge, runs through all of vector calculus.

GRE Mathematics vector-calculus questions ask you to:

- Integrate in polar, cylindrical and spherical coordinates with the right Jacobian.
- Compute the **divergence** 散度 and curl of a field.
- Decide whether a field is a **conservative field** 保守场.
- Apply Green's, Stokes' and the divergence theorems with the right orientation.

Coordinate changes

- Polar: $dA = r\,dr\,d\theta$. Cylindrical: $dV = r\,dr\,d\theta\,dz$.
- Spherical, with ϕ measured from the positive z-axis: $dV = \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$.
- A ball of radius R: $\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \frac{4}{3}\pi R^3$.



Spherical coordinates and their Jacobian.

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Divergence and curl

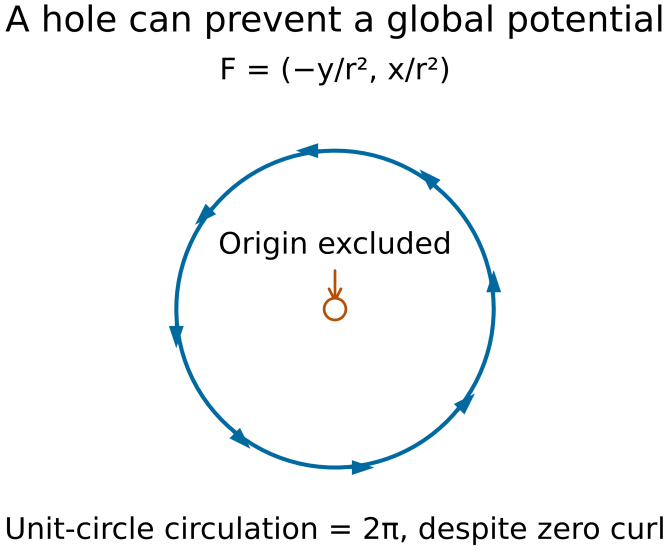
For $F = (P, Q, R)$:

- $\operatorname{div} F = P_x + Q_y + R_z$, the outflow per unit volume.
- $\operatorname{curl} F = \nabla \times F$, the circulation per unit area.
- $\operatorname{curl}(\nabla f) = 0$ and $\operatorname{div}(\operatorname{curl} F) = 0$ for smooth fields.

Conservative fields

$F = \nabla f$ for some potential f exactly when line integrals of F depend only on the endpoints.

- A conservative field has zero curl.
- Zero curl implies conservative only on a simply connected region, one with no holes.
- $F = \left(\dfrac{-y}{x^2 + y^2}, \dfrac{x}{x^2 + y^2} \right)$ has zero curl away from the origin, but its circulation around the unit circle is 2π.



Zero curl but circulation 2π: the hole at the origin prevents a global potential

The integral theorems

$$\oint_{\partial S} F \cdot dr = \iint_S (\nabla \times F) \cdot dS, \qquad \iint_{\partial V} F \cdot dS = \iiint_V \nabla \cdot F \, dV$$

- Green's theorem is the flat case of Stokes' theorem: $\oint P\,dx + Q\,dy = \iint (Q_x - P_y)\,dA$.
- Orientation: the boundary curve runs anticlockwise when seen from the side the normal points to; closed surfaces use the outward normal.

- The field must be smooth on the whole region, holes included.

Worked example: an annulus and a sphere

- $\iint (x^2 + y^2)\, dA$ over $1 \le x^2 + y^2 \le 4$: $\displaystyle\int_0^{2\pi} \int_1^2 r^3\, dr\, d\theta = 2\pi \times \frac{16 - 1}{4} = \frac{15\pi}{2}$.
- Outward flux of $F = (x, y, z)$ through the unit sphere: $\operatorname{div} F = 3$, so the flux is $3 \times \frac{4}{3}\pi = 4\pi$.
- Check directly: on the sphere $F \cdot n = 1$, and the area is 4π.

Worked example: a field with a hole

$F = \left(\dfrac{-y}{x^2 + y^2}, \dfrac{x}{x^2 + y^2} \right)$.

- $Q_x - P_y = 0$ wherever $(x, y) \neq 0$.
- On $x = \cos t$, $y = \sin t$: $F \cdot dr = (\sin^2 t + \cos^2 t)\, dt$, so the circulation is 2π.
- Green's theorem cannot be used on the disc, because F is undefined at the origin. On a region that does not surround the origin, the circulation is 0.

Common errors

- Writing $dV = \rho \sin \phi$ or forgetting the Jacobian.
- Assuming zero curl means conservative on a region with a hole.
- Using an inward normal without changing the sign.
- Applying a theorem where the field is not defined.
- Mixing up the roles of ϕ and θ.

Key terms

divergence: $P_x + Q_y + R_z$, a field's net outflow per unit volume.

conservative field: A field that is the gradient of a potential, so its line integrals depend only on the endpoints.

Check yourself

- What is the divergence of (x^2, y^2, z^2)?
- What is the outward flux of $(0, 0, z)$ through the unit sphere?
- Is $(2xy, x^2)$ conservative on the plane? If so, give a potential.

Answers: $2x + 2y + 2z$. $\frac{4}{3}\pi$. Yes: $x^2 y$.

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T.6 · Functions, inverse branches and composition

GRE Subject Test · GRE Mathematics

Functions, inverse branches and composition

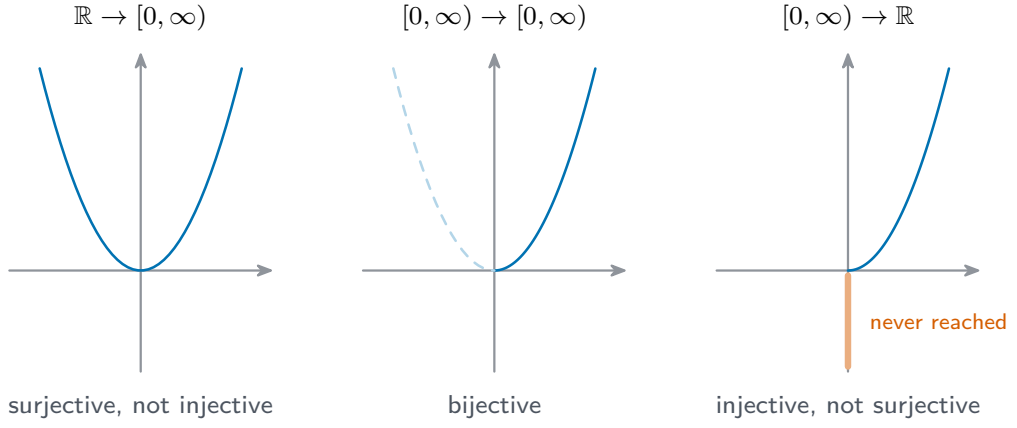
In 1837 Peter Gustav Lejeune Dirichlet gave the modern idea of a function: any rule that assigns to each input exactly one output, whether or not a formula describes it. Earlier, in 1829, he had already given a startling example: the function that is 1 at every rational number and 0 at every irrational one. It has no graph that can be drawn and is continuous nowhere. A function, on this view, is defined by its domain, its codomain and its rule, and changing any of them changes the function.

GRE Mathematics function questions ask you to:

- Decide whether a function is injective, surjective or a **bijection** 双射, with stated domain and codomain.
- Restrict a function to a branch and find its inverse.
- Use inverse trigonometric functions with their correct ranges.
- Reason about compositions and **involutions** 对合.

Injective, surjective, bijective

- Injective (one-to-one): different inputs give different outputs.
- Surjective (onto): every element of the codomain is an output.
- Bijective: both. Only a bijection has an inverse function.
- The same rule x^2 is surjective but not injective from $\mathbb{R}$ to $[0,\infty)$, bijective from $[0,\infty)$ to $[0,\infty)$, and injective but not surjective from $[0,\infty)$ to $\mathbb{R}$.



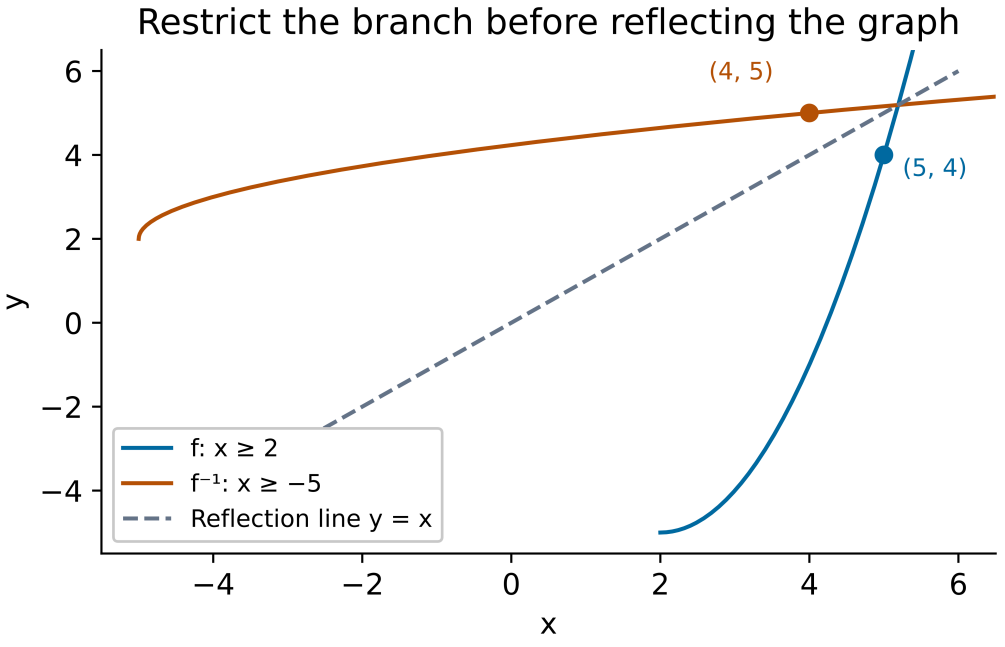
Same rule, different domain and codomain.

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Inverse branches

To invert a function that is not one-to-one, restrict it to a branch on which it is.

- $\sqrt{y}$ is the inverse of x^2 on $x \geq 0$.
- arcsin has range $[-\pi/2, \pi/2]$, arccos has $[0, \pi]$, arctan has $(-\pi/2, \pi/2)$.
- $\arcsin(\sin x) = x$ only when x is in $[-\pi/2, \pi/2]$; $\sin(\arcsin y) = y$ for every y in $[-1, 1]$.
- The graph of f^{-1} is the reflection of the graph of f in $y = x$.



Restrict to one branch before reflecting: (5, 4) on f becomes (4, 5) on f^{-1}

Composition

- $(g \circ f)(x) = g(f(x))$: apply f first. Usually $g \circ f \neq f \circ g$.
- $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
- If $g \circ f$ is injective, then f is injective. If $g \circ f$ is surjective, then g is surjective.
- An involution is its own inverse: $f(f(x)) = x$. Examples: $1/x$ on $x \neq 0$, $a - x$, and reflection in a line.

Worked example: a branch

Restrict $f(x) = (x-1)^2 + 2$ to $x \geq 1$.

- The image is $[2, \infty)$.
- Solving $y = (x-1)^2+2$ with $x \geq 1$ gives $x = 1+\sqrt{y-2}$, so $f^{-1} : [2,\infty) \to [1,\infty)$, $f^{-1}(y) = 1 + \sqrt{y-2}$.
- Check: $f^{-1}(f(x)) = 1 + |x-1| = x$ because $x \geq 1$; $f(f^{-1}(y)) = y$.

Worked example: compositions

- $\arcsin\left(\sin\frac{5\pi}{6}\right) = \arcsin\frac{1}{2} = \frac{\pi}{6}$, not $\frac{5\pi}{6}$.
- $f(x) = \dfrac{x+1}{x-1}$: $f(f(x)) = \dfrac{2x/(x-1)}{2/(x-1)} = x$, so f is an involution on $x \neq 1$.
- If $f(x) = 2x+1$ and $g(x) = x^3$, then $(g \circ f)^{-1}(y) = \dfrac{\sqrt[3]{y}-1}{2}$.

Common errors

- Calling a function bijective without stating the codomain.
- Taking both square-root branches as one inverse.
- Using $\arcsin(\sin x) = x$ outside $[-\pi/2, \pi/2]$.
- Reversing the order in $(g \circ f)^{-1}$.
- Assuming composition is commutative.

Key terms

bijection: A function that is both injective and surjective between its stated sets.

involutions: Functions that are their own inverses: $f(f(x)) = x$.

Check yourself

- Is e^x a bijection from $\mathbb{R}$ to $\mathbb{R}$?
- What is $\arccos(\cos(-\pi/3))$?
- What is the inverse of $f(x) = \dfrac{2x+1}{x-3}$?

Answers: no: it is injective but not onto $\mathbb{R}$ (it misses $(-\infty, 0]$). $\frac{\pi}{3}$. $\dfrac{3y+1}{y-2}$.

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T.7 · Set images, equivalence relations and logical negation

GRE Subject Test · GRE Mathematics

Set images, equivalence relations and logical negation

In June 1902 Gottlob Frege was finishing the second volume of his life's work, a foundation of all arithmetic on logic, when a letter arrived from the young Bertrand Russell. Consider the set of all sets that are not members of themselves, Russell wrote. Is it a member of itself? If it is, it is not; if it is not, it is. Frege's system allowed such a set and was therefore inconsistent. He added an appendix: "Hardly anything more unfortunate can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished." Precise rules about sets and logic grew from that crisis.

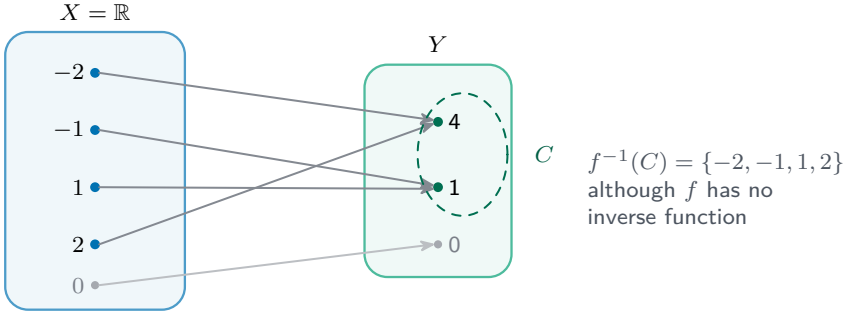
GRE Mathematics set and logic questions ask you to:

- Compute images and **preimages** 原像 and know which operations they preserve.
- Check the three properties of an **equivalence relation** 等价关系 and find its classes.
- Negate statements with quantifiers correctly.
- Separate the converse from the contrapositive.

Images and preimages

For $f : X \to Y$, the image of A is $f(A) = \{f(a) : a \in A\}$; the preimage of C is $f^{-1}(C) = \{x : f(x) \in C\}$.

- A preimage needs no inverse function: for $f(x) = x^2$, $f^{-1}(\{1, 4\}) = \{-2, -1, 1, 2\}$.
- Preimages preserve everything: unions, intersections and complements.
- Images preserve unions, but $f(A \cap B) \subseteq f(A) \cap f(B)$ can be strict. Equality for all sets holds exactly when f is injective.



A preimage needs no inverse function.

Equivalence relations

A relation $\sim$ is an equivalence relation when it is:

- Reflexive: $a \sim a$. Symmetric: $a \sim b$ implies $b \sim a$. Transitive: $a \sim b$ and $b \sim c$ imply $a \sim c$.
- Its classes partition the set, and every partition comes from an equivalence relation.
- $a \sim b$ when $3 \mid (a - b)$, on $\mathbb{Z}$, has three classes: the residues 0, 1 and 2 mod 3.
- $\leq$ is reflexive and transitive but not symmetric, so it is not an equivalence relation.

Negation

- $\neg(\forall x\, P(x))$ is $\exists x\, \neg P(x)$; $\neg(\exists x\, P(x))$ is $\forall x\, \neg P(x)$.
- $\neg(P \wedge Q)$ is $\neg P \vee \neg Q$; $\neg(P \Rightarrow Q)$ is $P \wedge \neg Q$.
- Negate from the outside in, flipping each quantifier and keeping its order.

Conditionals

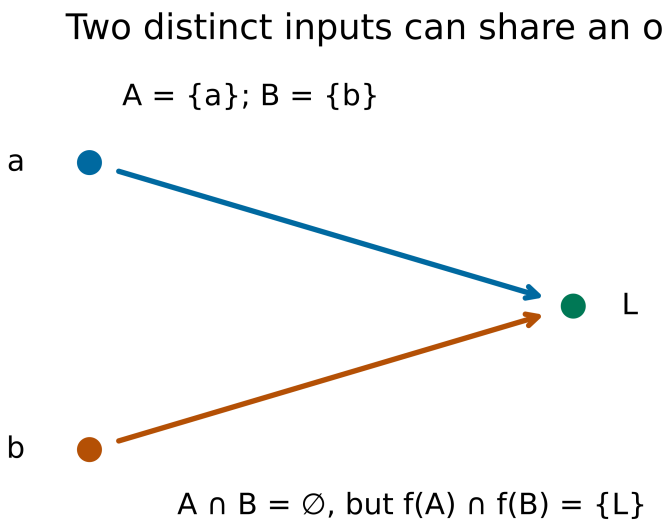
- $P \Rightarrow Q$ is equivalent to its contrapositive $\neg Q \Rightarrow \neg P$.
- The converse $Q \Rightarrow P$ and the inverse $\neg P \Rightarrow \neg Q$ are different statements.
- "Differentiable implies continuous" is true; its converse is false ($|x|$).

Worked example: images of intersections

Prove $f(A \cap B) \subseteq f(A) \cap f(B)$, and show it can be strict.

- If $y = f(x)$ with $x \in A \cap B$, then $y \in f(A)$ and $y \in f(B)$.
- Strict: let $f(a) = f(b) = c$ with $a \neq b$, $A = \{a\}$, $B = \{b\}$. Then $f(A \cap B) = \varnothing$ but $f(A) \cap f(B) = \{c\}$.
- If f is injective and $y = f(a) = f(b)$ with $a \in A$, $b \in B$, then $a = b \in A \cap B$, so equality holds.

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f(A ∩ B) can be empty while f(A) ∩ f(B) is not: images do not preserve intersections

Worked example: negating continuity

f is continuous at a: $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x\ \ (|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon)$.

- Negation: $\exists \varepsilon > 0\ \forall \delta > 0\ \exists x\ \ (|x - a| < \delta \wedge |f(x) - f(a)| \geq \varepsilon)$.
- In words: some tolerance cannot be met, however close you look.
- "Every student passed some test" negates to "some student failed every test".

Common errors

- Assuming images preserve intersections.
- Thinking a preimage requires an inverse function.
- Negating "for all" as "for none".
- Treating the converse as equivalent to the statement.
- Calling a relation an equivalence after checking only two properties.

Key terms

preimages: Sets of all inputs whose outputs lie in a given set.

equivalence relation: A relation that is reflexive, symmetric and transitive.

Check yourself

- For $f(x) = x^2$, what is $f^{-1}([4, 9])$?
- How many equivalence relations are there on a set with 3 elements?
- What is the negation of "$P \Rightarrow Q$"?

Answers: $[-3, -2] \cup [2, 3]$. Five (one for each partition). P and not Q.

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T.8 · Conditioning, Bayesian inference and sampling error

GRE Subject Test · GRE Mathematics

Conditioning, Bayesian inference and sampling error

In 1763, two years after the death of Thomas Bayes, a Presbyterian minister in Kent, his friend Richard Price read Bayes's unpublished essay to the Royal Society in London. It answered a reversed question: given how often an event has happened, how probable is it that its chance lies in a certain range? Bayes's rule for turning $P(\text{evidence} \mid \text{cause})$ into $P(\text{cause} \mid \text{evidence})$ was little used for almost two centuries. Today it runs spam filters, medical tests and much of machine learning.

GRE Mathematics probability and statistics questions ask you to:

- Compute conditional probabilities and use the law of total probability.
- Find a **posterior probability** 后验概率 with Bayes' rule.
- Separate independence from mutual exclusion.
- Use expectation and variance of sums, and the **standard error** 标准误 of a mean.

Conditional probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \qquad P(B) = \sum_{i} P(B \mid A_i) P(A_i)$$

- The second formula, the law of total probability, needs the A_i to split the whole space.
- $P(A \mid B)$ and $P(B \mid A)$ are usually different.

Bayes' rule

$$P(A \mid B) = \frac{P(B \mid A) P(A)}{P(B)}$$

- The prior $P(A)$ is the starting probability; the posterior $P(A \mid B)$ is updated by the evidence.
- Expected counts make it concrete: imagine 1,000 cases and fill in a table.
- A rare cause stays fairly unlikely after one positive test unless the test rarely gives false positives.

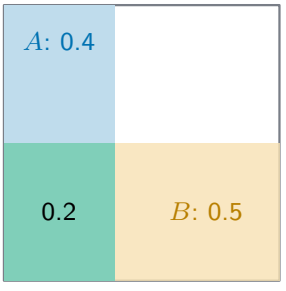
Condition on positives: 80 / 125, not 80 / 100

Expected group	Positive	Negative	Total
Defective	80	20	100
Not defective	45	855	900
Total	125	875	1000

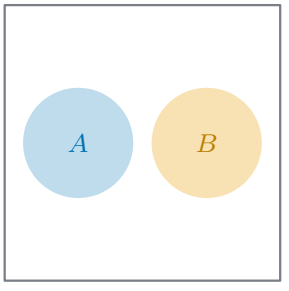
Given a positive test, divide by all 125 positives: 80/125, not 80/100

Independent and exclusive

- Independent: $P(A \cap B) = P(A)P(B)$. Knowing A tells you nothing about B.
- Mutually exclusive: $P(A \cap B) = 0$.
- Two events with positive probabilities cannot be both: if they are exclusive, knowing A tells you B did not happen.



independent
$P(A \cap B) = 0.4 \times 0.5$



mutually exclusive
$P(A \cap B) = 0 \neq P(A)P(B)$

Independent is not the same as mutually exclusive.

Sums and sampling error

- $E[aX + bY] = aE[X] + bE[Y]$ always. $\operatorname{Var}(X + Y) = \operatorname{Var} X + \operatorname{Var} Y$ when X and Y are independent.
- The mean of n independent observations with standard deviation σ has standard error $\sigma/\sqrt{n}$.

- Central limit theorem: for large n the sample mean is approximately normal. A 95% interval is about the estimate ± 1.96 standard errors.
- To halve the standard error, quadruple the sample.

Worked example: a defect test

10% of devices are defective. A test is positive for 80% of defective devices and 5% of good ones.

- In 1,000 devices: 100 defective, 80 positive; 900 good, 45 positive.
- $P(\text{defect} \mid +) = \dfrac{80}{80 + 45} = \dfrac{16}{25} = 0.64$.
- The 0.8 is $P(+ \mid \text{defect})$, the reversed condition. Check: the negatives are $20 + 855 = 875$, and $125 + 875 = 1000$.

Worked example: sampling error

Test scores have $\sigma = 10$. A sample of 25 has mean 72.

- Standard error: $10/\sqrt{25} = 2$.
- An approximate 95% interval is $72 \pm 1.96 \times 2 = 72 \pm 3.9$.
- With 100 students the standard error would be 1: four times the sample for half the error.

Common errors

- Confusing $P(A \mid B)$ with $P(B \mid A)$.
- Ignoring the prior (base rate) in Bayes' rule.
- Treating exclusive events as independent.
- Adding standard deviations instead of variances.
- Thinking the standard error falls in proportion to n.

Key terms

posterior probability: The probability of a cause after taking account of the observed evidence.

standard error: The standard deviation of a statistic, such as a sample mean, across repeated samples.

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Check yourself

- A and B are independent with $P(A) = 0.3$, $P(B) = 0.5$. What is $P(A \cup B)$?
- A family has two children, at least one a girl. What is the probability both are girls?
- Observations have SD 8. What is the standard error of the mean of 16 of them?

Answers: $0.3 + 0.5 - 0.15 = 0.65$. $\frac{1}{3}$. 2.

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T.9 · Similarity, scaling and conic distance loci

GRE Subject Test · GRE Mathematics

Similarity, scaling and conic distance loci

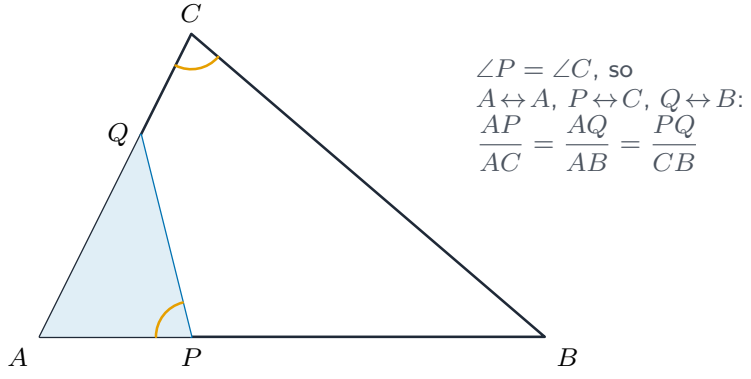
Around 240 BC Eratosthenes, librarian at Alexandria, heard that at noon on midsummer's day the Sun shone straight down a well at Syene, far to the south. At the same moment in Alexandria, a vertical post cast a shadow at an angle of about 7.2°, one fiftieth of a circle. With the Sun's rays parallel, that angle equals the angle at the Earth's centre between the two cities. The distance between them, about 5,000 stadia, was therefore one fiftieth of the Earth's circumference: about 250,000 stadia. **Similarity** 相似 and parallel lines measured a planet.

GRE Mathematics geometry questions ask you to:

- Set up similar triangles with the correct vertex correspondence.
- Scale lengths, areas and volumes.
- Use the distance definitions of the conics, each with a **focus** 焦点.
- Find the axes, foci and eccentricity from an equation.

Similar triangles

- Two triangles are similar if two pairs of angles are equal (AA). Then all corresponding sides are in one ratio.
- Write the correspondence first: in $\triangle APQ \sim \triangle ACB$, $A \leftrightarrow A$, $P \leftrightarrow C$, $Q \leftrightarrow B$, so $\dfrac{AP}{AC} = \dfrac{AQ}{AB} = \dfrac{PQ}{CB}$.
- A line parallel to one side of a triangle cuts off a similar triangle.



Write the vertex correspondence before the ratios.

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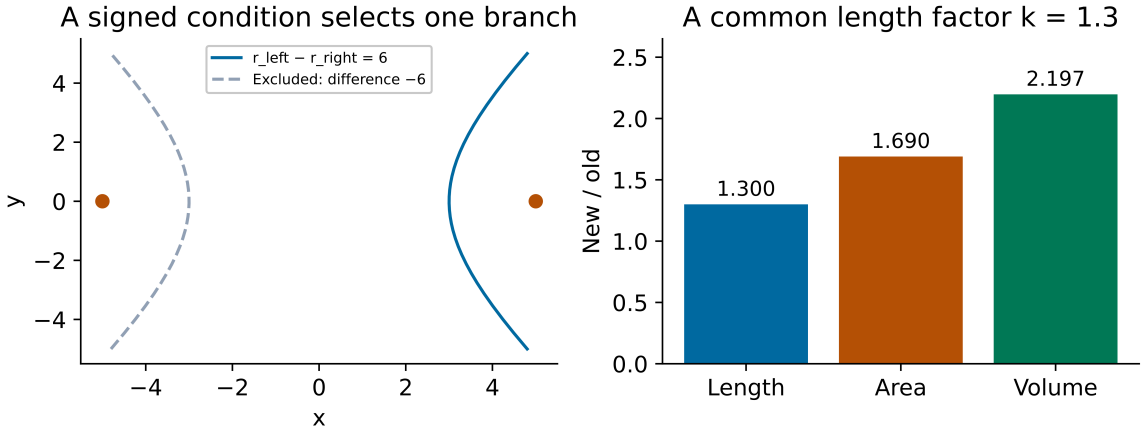
Scaling

If lengths scale by k:

- Areas scale by k^2 and volumes by k^3.
- An area ratio of $9:25$ means a length ratio of $3:5$ and a volume ratio of $27:125$.
- An area ratio alone does not prove that two solids are similar.

Conics as loci

- Ellipse: the sum of distances to two foci is $2a$. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, $c^2=a^2-b^2$, foci $(\pm c,0)$.
- Hyperbola: the difference of distances is $\pm 2a$. $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$, $c^2=a^2+b^2$. A signed difference picks one branch.
- Parabola: distance to the focus equals distance to the directrix. Focus $(0,p)$ and directrix $y=-p$ give $x^2=4py$.
- Eccentricity $e=c/a$: below 1 for an ellipse, 1 for a parabola, above 1 for a hyperbola.



A signed difference picks one branch; a length factor k scales area by k^2 and volume by k^3

Worked example: scale factors

- Similar solids with surface areas $9:25$ have lengths $3:5$ and volumes $27:125$.
- A model at scale $1:50$ has $1/2500$ of the real area and $1/125000$ of the volume.
- A 1.3 times enlargement multiplies area by 1.69 and volume by 2.197.

Worked example: finding the conic

- Points with $r_{\text{left}} - r_{\text{right}} = 6$ for foci $(\pm 5, 0)$: $a = 3$, $b^2 = 25 - 9 = 16$, so $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$. Only the right branch, $x \geq 3$, is closer to the right focus.
- Foci $(\pm 3, 0)$ and distance sum 10: $a = 5$, $b^2 = 25 - 9 = 16$, the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$, with $e = 0.6$.
- Focus $(0, 2)$ and directrix $y = -2$: $x^2 = 8y$.

Common errors

- Writing ratios before matching equal angles.
- Scaling area or volume by k instead of k^2 or k^3.
- Using $c^2 = a^2 - b^2$ for a hyperbola.
- Giving both branches when the difference has a sign.
- Confusing the focus with the vertex.

Key terms

similarity: Having equal corresponding angles and one common ratio of corresponding lengths.

focus: A fixed point used to define a conic by distances.

Check yourself

- Two similar cones have volumes $8 : 27$. What is the ratio of their heights?
- What is the eccentricity of $\dfrac{x^2}{16} + \dfrac{y^2}{7} = 1$?
- Where is the focus of $y^2 = 12x$?

Answers: $2 : 3$. $c = 3$, so $e = \frac{3}{4}$. $(3, 0)$.

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C.8 · Trigonometric phase and parametric curves

GRE Subject Test · GRE Mathematics

Trigonometric phase and parametric curves

In 1659 Christiaan Huygens, in The Hague, discovered a remarkable property of the cycloid, the curve traced by a point on the rim of a rolling wheel. A bead sliding down an upside-down cycloid reaches the bottom in the same time wherever it starts. Huygens used this to design a pendulum clock whose swing takes the same time whether it is large or small, and described it in 1673. The cycloid is easiest to describe as a **parametric curve** 参数曲线: $x = r(t - \sin t)$, $y = r(1 - \cos t)$.

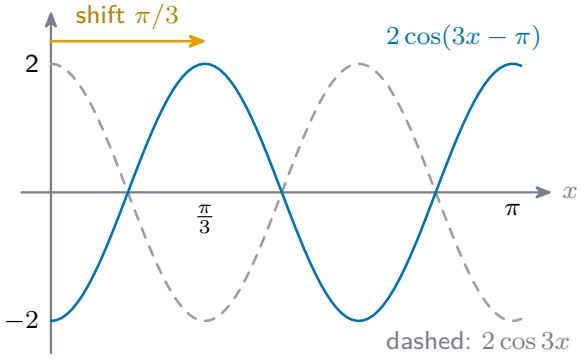
GRE Mathematics trigonometry and parametric questions ask you to:

- Read amplitude, period, **phase angle** 相位角 and horizontal shift.
- Combine $a\cos x + b\sin x$ into one wave.
- Parametrise curves and find slopes, lengths and areas from a parametrisation.

Sinusoids

For $y = A\cos(\omega x - \varphi)$:

- Amplitude $|A|$, period $2\pi/\omega$.
- The phase is φ, but the horizontal shift is φ/ω, to the right. They are equal only when $\omega = 1$.
- A negative coefficient is a phase of π: $-3\cos 2x = 3\cos(2x - \pi)$.



Horizontal shift = phase / omega.

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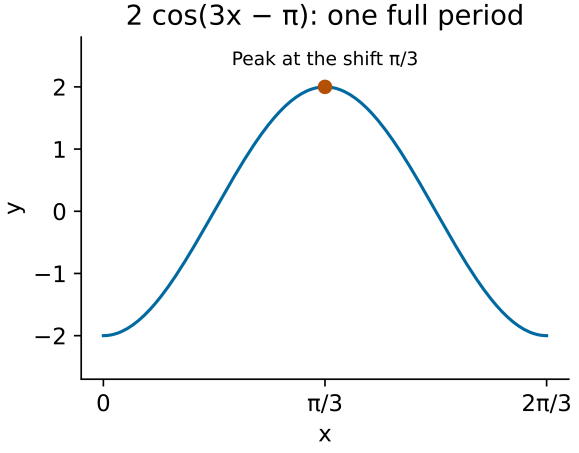
Combining waves

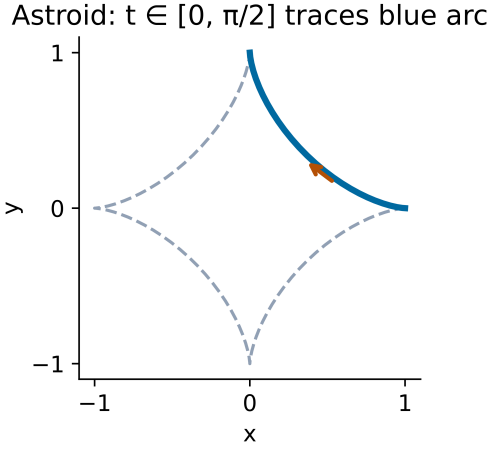
$$a\cos x + b\sin x = R\cos(x-\alpha), \qquad R=\sqrt{a^2+b^2}, \quad \cos\alpha=\frac{a}{R},\ \sin\alpha=\frac{b}{R}$$

- The maximum is R and the minimum $-R$.
- Use both $\cos\alpha$ and $\sin\alpha$ to place α in the right quadrant; $\tan\alpha = b/a$ alone is ambiguous.

Parametric curves

- Circle: $(r\cos t, r\sin t)$. Ellipse: $(a\cos t, b\sin t)$. Both run anticlockwise as t increases.
- Slope: $\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}$, where $dx/dt\neq 0$.
- Arc length: $\displaystyle\int \sqrt{x'(t)^2+y'(t)^2}\,dt$.
- Area under a curve: $\displaystyle\int y\,\frac{dx}{dt}\,dt$, with the limits in the direction of increasing x.





The phase shift $\pi/3$ moves the peak; the period is $2\pi/3$

Worked example: reading a wave

$y=-3\cos 2x$.

- Amplitude 3, period π.
- $y=3\cos(2x-\pi)$: phase π, shift $\pi/2$ to the right.
- $3\sin x+4\cos x = 5\cos(x-\alpha)$ with $\cos\alpha=\frac{4}{5}$, $\sin\alpha=\frac{3}{5}$, so $\alpha=0.644$. Its maximum is 5.

Worked example: curves

- The ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$: $x = 3\cos t$, $y = 2\sin t$, starting at $(3, 0)$ and moving anticlockwise.
- One arch of the cycloid ($0 \le t \le 2\pi$): $x' = r(1 - \cos t)$, $y' = r\sin t$, so the speed is $2r\sin(t/2)$ and the length is $\displaystyle\int_0^{2\pi} 2r\sin\frac{t}{2}\,dt = 8r$.
- The astroid $x = \cos^3 t$, $y = \sin^3 t$ has total length 6.

Common errors

- Reading the phase as the horizontal shift.
- Forgetting that a negative amplitude is a phase of π.
- Placing α in the wrong quadrant from $\tan\alpha$ alone.
- Dividing dx/dt by dy/dt for the slope.
- Running the area integral in the wrong direction and losing the sign.

Key terms

parametric curve: A curve whose coordinates are given as functions of a shared parameter.

phase angle: The angle added inside a periodic function's argument, defined up to a full period.

Check yourself

- What is the period of $\sin 4x$?
- Which way, and how far, is $\cos(2x + \pi/2)$ shifted from $\cos 2x$?
- What is the slope of $x = \cos t$, $y = \sin t$ at $t = \pi/4$?

Answers: $\pi/2$. Left by $\pi/4$. -1.

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T.10 · Vector geometry, projections and oriented area

GRE Subject Test · GRE Mathematics

Vector geometry, projections and oriented area

On 16 October 1843 William Rowan Hamilton was walking along the Royal Canal in Dublin when he suddenly saw how to multiply three-dimensional quantities. He cut the formula into the stone of Brougham Bridge with his knife: $i^2 = j^2 = k^2 = ijk = -1$. Multiplying two of his quaternions with no real part gives two pieces: one is minus the dot product, the other is the cross product. Later physicists kept these two products and dropped the rest.

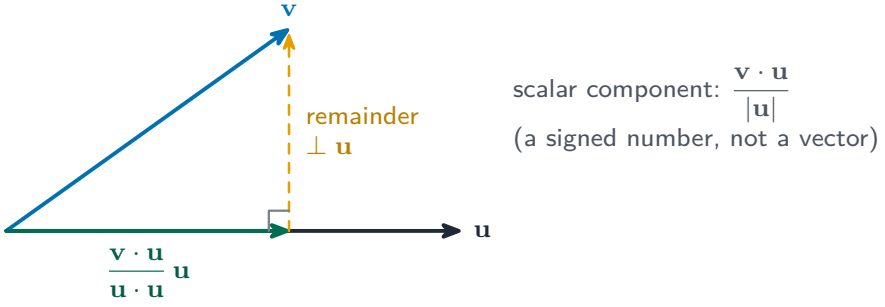
GRE Mathematics vector questions ask you to:

- Use the dot product for angles, **orthogonal** 正交 vectors and projections.
- Use the **cross product** 叉积 for areas, normals and orientation.
- Use the triple product for volumes.
- Write lines and planes, and find distances.

The dot product

$u \cdot v = u_1v_1 + u_2v_2 + u_3v_3 = |u||v| \cos\theta$.

- $u \cdot v = 0$ exactly when u and v are orthogonal.
- The scalar component of v along u is $\dfrac{u \cdot v}{|u|}$, a signed number.
- The vector projection is $\operatorname{proj}_u v = \dfrac{u \cdot v}{u \cdot u} u$. The remainder $v - \operatorname{proj}_u v$ is orthogonal to u.



Projection and orthogonal remainder.

The cross product

$u \times v = (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1)$.

- It is perpendicular to both u and v, with direction given by the right-hand rule.
- $|u \times v| = |u||v| \sin\theta$, the area of the parallelogram on u and v. A triangle has half that area.
- $v \times u = -(u \times v)$: the order sets the orientation.

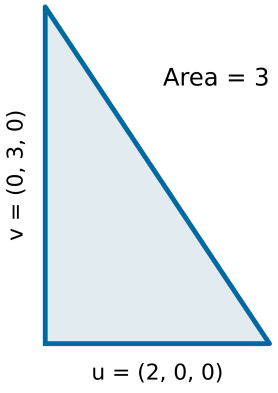
Volumes

- The triple product $u \cdot (v \times w)$ equals the determinant with rows u, v, w.
- Its absolute value is the volume of the parallelepiped; it is zero exactly when the three vectors lie in one plane.

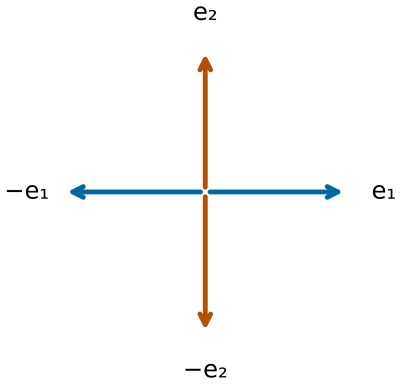
Lines and planes

- A line: $r = r_0 + tv$.
- A plane with normal $n = (a, b, c)$: $ax + by + cz = d$.
- Distance from (x_0, y_0, z_0) to that plane: $\dfrac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$.

Triangle area = |u × v| / 2 = 3



Two negative products; four zeros



Triangle area = |u × v| / 2 = 6 / 2 = 3

Worked example: projection and area

$u = (1, 0, 2)$, $v = (2, 3, 6)$.

- $u \cdot v = 14$ and $u \cdot u = 5$, so $\operatorname{proj}_u v = \frac{14}{5}(1, 0, 2) = (2.8, 0, 5.6)$.
- The remainder is $(-0.8, 3, 0.4)$, and its dot product with u is $-0.8 + 0.8 = 0$.
- $u \times v = (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1) = (0-6,\ 4-6,\ 3-0) = (-6, -2, 3)$, of length $\sqrt{36 + 4 + 9} = 7$, so the triangle O, u, v has area $\frac{7}{2}$.

Worked example: distance and volume

- Distance from $(1, 2, 3)$ to $x + 2y + 2z = 4$: $\dfrac{|1 + 4 + 6 - 4|}{3} = \dfrac{7}{3}$.
- The parallelepiped on $(1, 0, 0)$, $(1, 2, 0)$, $(1, 1, 3)$ has volume $|\det| = 1 \times (2 \times 3 - 0) = 6$.
- The angle between $(1, 1, 0)$ and $(0, 1, 1)$: $\cos\theta = \dfrac{1}{\sqrt{2}\sqrt{2}} = \dfrac{1}{2}$, so $60°$.

Common errors

- Confusing the scalar component with the vector projection.
- Dividing by $|u|$ instead of $u \cdot u$ in the vector projection.
- Reversing the order in a cross product and losing the orientation.
- Forgetting the factor $\frac{1}{2}$ for a triangle.
- Dropping the absolute value in the distance formula.

Key terms

orthogonal: Having zero dot product.

cross product: The vector $u \times v$, perpendicular to u and v, whose length is the area of the parallelogram they span.

Check yourself

- Are $(1, 2, -1)$ and $(3, -1, 1)$ orthogonal?
- What is $(1, 0, 0) \times (0, 1, 0)$?
- What is the area of the triangle with vertices $(0, 0, 0)$, $(2, 0, 0)$ and $(0, 3, 0)$?

Answers: yes: $3 - 2 - 1 = 0$. $(0, 0, 1)$. 3.

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C.9 · Integrating factors and nonhomogeneous differential equations

GRE Subject Test · GRE Mathematics

Integrating factors and nonhomogeneous differential equations

On 10 June 2000 thousands of people walked across London's new Millennium Bridge on its opening day. The bridge began to sway from side to side. Walkers instinctively adjusted their steps to the sway, which pushed it further: the bridge's natural sideways frequency, close to 1 Hz, matched the side-to-side rhythm of walking. It closed after two days and reopened in 2002, fitted with dampers. In a differential equation, a forcing term that matches a natural solution produces a response that keeps growing, and the method for finding it changes too.

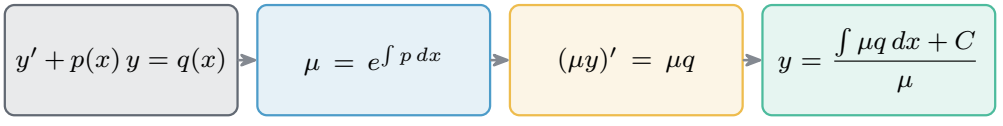
GRE Mathematics questions on nonhomogeneous equations ask you to:

- Solve first-order linear equations with an **integrating factor** 积分因子.
- Write general solutions as homogeneous plus **particular solution** 特解.
- Choose trial solutions by undetermined coefficients, including the resonance case.
- Check solutions and use initial conditions.

First-order linear equations

For $y' + p(x)y = q(x)$:

- Multiply by $\mu = e^{\int p\,dx}$. The left side becomes $(\mu y)'$.
- Integrate: $\mu y = \int \mu q\,dx + C$, so $y = \dfrac{1}{\mu}\left(\int \mu q\,dx + C\right)$.
- The constant must be included before dividing by μ.



Integrating factor in four steps.

Superposition

For a linear equation $L[y] = g$:

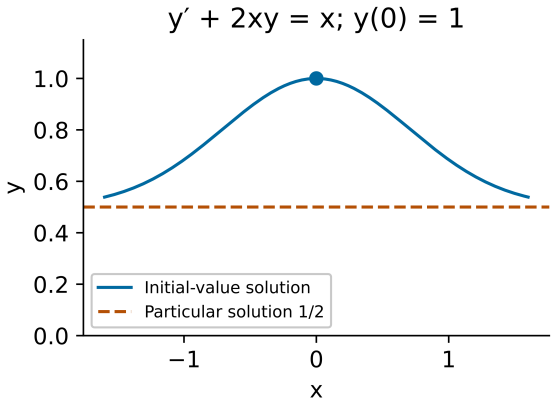
- The general solution is $y = y_h + y_p$: all solutions of $L[y] = 0$, plus one solution of $L[y] = g$.

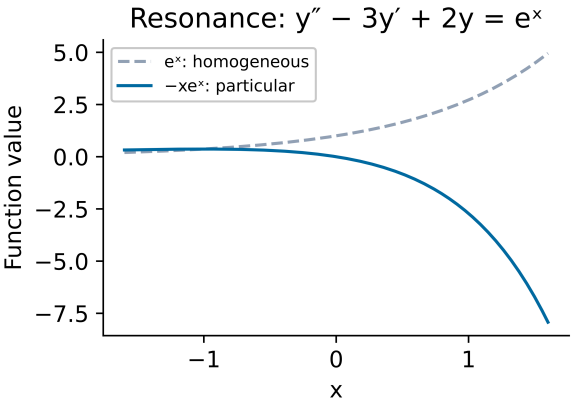
- If $g = g_1 + g_2$, add particular solutions for each part.
- Apply the initial conditions to the full y, not to y_h alone.

Undetermined coefficients

For constant coefficients, guess a particular solution of the same form as g:

- $g = e^{kx}$: try Ae^{kx}. g a polynomial of degree n: try a general polynomial of degree n.
- $g = \cos \omega x$ or $\sin \omega x$: try $A \cos \omega x + B \sin \omega x$.
- Resonance: if the trial already solves the homogeneous equation, multiply it by x (by x^2 for a double root).





When the forcing term solves the homogeneous equation, multiply the trial solution by x

Worked example: an integrating factor

Solve $y' + 2y = e^x$, $y(0) = 1$.

- $\mu = e^{2x}$, so $(e^{2x}y)' = e^{3x}$ and $e^{2x}y = \frac{1}{3}e^{3x} + C$.
- $y = \frac{1}{3}e^x + Ce^{-2x}$, and $y(0) = 1$ gives $C = \frac{2}{3}$.
- Check: $y' + 2y = \frac{1}{3}e^x - \frac{4}{3}e^{-2x} + \frac{2}{3}e^x + \frac{4}{3}e^{-2x} = e^x$.
- Similarly $y' + 2xy = x$, $y(0) = 1$ has $\mu = e^{x^2}$ and solution $y = \frac{1}{2} + \frac{1}{2}e^{-x^2}$, which tends to $\frac{1}{2}$.

Worked example: resonance

Find a particular solution of $y'' - 3y' + 2y = e^x$.

- The characteristic roots are 1 and 2, so e^x solves the homogeneous equation and Ae^x fails.
- Try $y = Axe^x$: $y' = A(x+1)e^x$, $y'' = A(x+2)e^x$, and $y'' - 3y' + 2y = A\left[(x+2) - 3(x+1) + 2x\right]e^x = -Ae^x$.
- So $A = -1$, $y_p = -xe^x$, and $y = C_1 e^x + C_2 e^{2x} - xe^x$.

Common errors

- Forgetting the constant of integration before dividing by μ.
- Using e^p instead of $e^{\int p\,dx}$.
- Using a trial solution that already solves the homogeneous equation.
- Applying initial conditions before adding y_p.
- Writing the equation with a coefficient on y' and not dividing by it first.

Key terms

integrating factor: A multiplier that turns a first-order linear equation into the derivative of a product.

particular solution: One solution of the nonhomogeneous equation, added to the homogeneous solutions.

Check yourself

- What is the integrating factor for $y' + y/x = 1$, $x > 0$?
- What trial solution suits $y'' + y = \sin x$?
- What is the general solution of $y' - y = 2$?

Answers: x. $x(A\cos x + B\sin x)$. $y = -2 + Ce^x$.

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T.11 · Weak compositions, loop invariants and flowchart tracing

GRE Subject Test · GRE Mathematics

Weak compositions, loop invariants and flowchart tracing

In 1969 Tony Hoare, a British computer scientist, published "An Axiomatic Basis for Computer Programming", building on work by Robert Floyd. Its central idea was simple: to prove a loop correct, find a statement that is true before the loop starts and stays true after every pass. When the loop stops, that statement, together with the stopping condition, gives the answer. This **loop invariant** 循环不变量 turns tracing a program into a short proof. Hoare later received the Turing Award.

GRE Mathematics counting and algorithm questions ask you to:

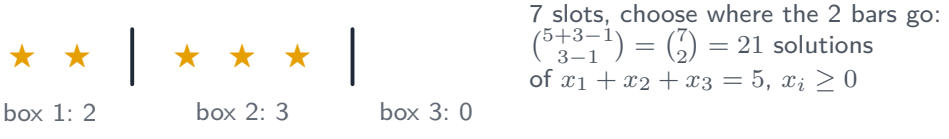
- Count each **weak composition** 弱组合分拆 with stars and bars.
- Handle lower bounds by shifting and upper bounds by inclusion–exclusion.
- Trace loops and flowcharts, counting iterations.
- State and use a loop invariant.

Stars and bars

The number of solutions of $x_1 + \cdots + x_k = n$ in nonnegative integers is

$$\binom{n+k-1}{k-1}$$

- Picture n stars and $k-1$ bars in a row; the bars split the stars into k groups.
- Positive integers (each $x_i \geq 1$): $\binom{n-1}{k-1}$.
- Lower bounds: substitute $x_i = u_i + m_i$ to remove each bound $x_i \geq m_i$.
- Upper bounds: count all solutions, then subtract those that break a bound, with inclusion–exclusion.



Stars and bars: C(n + k - 1, k - 1).

Tracing algorithms

- Make a table with one column per variable and one row per pass, and update the variables in the order the code does.
- Check the loop condition at the right moment: before the body for "while", after it for "repeat until".
- Count iterations carefully: off-by-one errors are the commonest mistake.

Loop invariants

A loop invariant is a statement that is:

- True before the first iteration.
- Kept true by every iteration.
- Useful at the end: combined with the stopping condition, it gives the result.

Odd-sum loop: S = m², B = 2m + 1 before the next addition

Iterations m	Sum S	Next addend B
0	0	1
1	1	3
2	4	5
3	9	7
4	16	9

The loop invariant $S = m^2$ holds before each addition

Worked example: bounded solutions

Count nonnegative solutions of $a + b + c = 8$ with $a \geq 2$.

- Put $u = a - 2$: $u + b + c = 6$ has $\binom{8}{2} = 28$ solutions.
- With also $b, c \geq 1$: subtract 1 from each to get a total of 4, so $\binom{6}{2} = 15$.
- So $28 - 15 = 13$ have $b = 0$ or $c = 0$. Check: $b = 0$ gives 7, $c = 0$ gives 7, both zero is counted twice: $7 + 7 - 1 = 13$.

Worked example: an odd-number loop

```
S = 0; B = 1
while B <= 2n - 1:
    S = S + B
    B = B + 2
output S
```

- Invariant after m passes: $S = m^2$ and $B = 2m + 1$. It holds at $m = 0$, and one pass gives $S = m^2 + 2m + 1 = (m + 1)^2$, $B = 2(m + 1) + 1$.
- The loop stops when $B = 2n + 1$, that is after n passes, so the output is n^2. For $n = 4$: 16.
- The loop `while x > 1: x = x // 2`, from $x = 100$, runs through 50, 25, 12, 6, 3, 1: six iterations.

Common errors

- Using $\binom{n+k-1}{k}$ with the wrong lower index, or mixing up nonnegative and positive solutions.
- Forgetting to shift lower bounds before counting.
- Counting a loop condition check as an extra pass.
- Stating an invariant that is true at the end but not kept by each pass.
- Updating two variables in the wrong order when tracing.

Key terms

weak composition: An ordered way of writing a number as a sum of nonnegative integers.

loop invariant: A statement that is true before and after every iteration of a loop.

Check yourself

- How many nonnegative solutions does $x + y + z = 10$ have?
- How many positive solutions does $a + b + c + d = 10$ have?
- What does the odd-number loop output for $n = 7$?

Answers: $\binom{12}{2} = 66$. $\binom{9}{3} = 84$. 49.

C.10 · Implicit differentiation and the inverse Jacobian

GRE Subject Test · GRE Mathematics

Implicit differentiation and the inverse Jacobian

In 1638 René Descartes challenged Pierre de Fermat to find the tangent line to a curve, now called the folium of Descartes: $x^3 + y^3 = 3axy$. Descartes had expected the problem to defeat Fermat's method of tangents. Fermat solved it easily. The curve cannot be written as one function $y = f(x)$, yet it has a tangent at almost every point. Implicit differentiation, and the theorems behind it, say when such a curve, or a coordinate change, can be treated locally as a function with an inverse.

GRE Mathematics implicit-function questions ask you to:

- Find dy/dx for a curve given implicitly.
- Know when the implicit function theorem applies.
- Compute a **Jacobian matrix** 雅可比矩阵 and use the inverse function theorem.
- Find the partial derivatives of a **local inverse** 局部逆映射.

Implicit differentiation

If $F(x, y) = 0$ defines y near a point, differentiate both sides with respect to x:

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

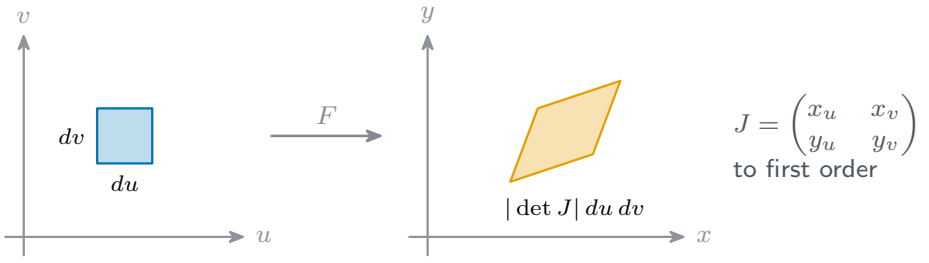
- This needs $F_y \neq 0$ at the point. The implicit function theorem guarantees a local function $y(x)$ there if F has continuous partial derivatives.
- Where $F_y = 0$, the tangent may be vertical, or the curve may cross itself.

The Jacobian matrix

For a map $(u, v) \mapsto (x, y)$:

$$J = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

- $|\det J|$ is the factor by which small areas are scaled.
- A small $du\,dv$ square is mapped to a parallelogram of area $|\det J|\,du\,dv$.



The Jacobian scales small areas.

The inverse function theorem

If the partial derivatives are continuous and $\det J \neq 0$ at a point, the map has a differentiable local inverse near that point, and

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = J^{-1}$$

- Invert the whole matrix. In general $u_x \neq 1/x_u$, because when x changes with y fixed, both u and v must move.
- $\det(J^{-1}) = 1/\det J$.

Coupled sensitivity: inverse entries differ from reciprocals

Forward map	Jacobian	Inverse Jacobian
x = u + v; y = u + 2v	[[1, 1], [1, 2]]	[[2, −1], [−1, 1]]
x = u² + v; y = u − v at (u, v) = (1, 0)	[[2, 1], [1, −1]]	[[1/3, 1/3], [1/3, −2/3]]

Invert the whole Jacobian matrix; its entries are not the reciprocals of the forward entries

Worked example: inverse partials

$x = u^2 + v$, $y = u - v$ at $(u, v) = (1, 0)$, where $(x, y) = (1, 1)$.

- $J = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$, with $\det J = -3 \neq 0$.
- $J^{-1} = \frac{1}{-3}\begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{pmatrix}$.
- So $u_x = \frac{1}{3}$, not $1/x_u = \frac{1}{2}$. Check: $JJ^{-1} = I$.

Worked example: the folium and polar coordinates

- For $x^3 + y^3 = 6xy$ at $(3, 3)$ (check: $54 = 54$): $F_x = 3x^2 - 6y = 9$ and $F_y = 3y^2 - 6x = 9$, so $dy/dx = -1$.
- Polar: $x = r\cos\theta$, so $x_\theta = -r\sin\theta$. But $\theta_x = -\sin\theta/r$, not $1/x_\theta = -1/(r\sin\theta)$.
- At the origin of the folium, $F_x = F_y = 0$: the curve crosses itself, and no single tangent exists.

Common errors

- Writing $u_x = 1/x_u$.
- Forgetting the minus sign in $dy/dx = -F_x/F_y$.
- Using the theorem where $F_y = 0$ or $\det J = 0$.
- Evaluating inverse partials at the input point's coordinates instead of matching input and output points.
- Forgetting the absolute value when the Jacobian scales areas.

Key terms

Jacobian matrix: The matrix of first partial derivatives of a map between spaces.

local inverse: An inverse map defined near one input point and its output.

Check yourself

- What is dy/dx on $x^2 + y^2 = 25$ at $(3, 4)$?
- A map has $\det J = 4$ at a point. What is the determinant of the inverse map's Jacobian there?
- Can $x^2 + y^2 = 1$ be written as $y = f(x)$ near $(1, 0)$?

Answers: $-\frac{3}{4}$. $\frac{1}{4}$. No: $F_y = 0$ there, and the tangent is vertical.

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A.7 · LU factorisation and nullity of composed maps

GRE Subject Test · GRE Mathematics

LU factorisation and nullity of composed maps

In 1948 Alan Turing, working on the first electronic computers in Britain, wrote a paper on how rounding errors grow when large linear systems are solved by machine. In it he described Gaussian elimination as splitting a matrix into a lower triangular factor and an upper triangular one, and named it the LU decomposition. Factor once, and every new right-hand side can be solved quickly by two easy substitutions. The method, with row swaps for stability, is still how computers solve linear systems.

GRE Mathematics questions on factorisations and composed maps ask you to:

- Find and use an LU factorisation, with **forward substitution** 前代 and back substitution.
- Place the permutation correctly when $PA = LU$.
- Read the determinant from the factors.
- Bound the rank and **nullity** 零度 of a composition.

LU factorisation

Gaussian elimination without row swaps writes $A = LU$:

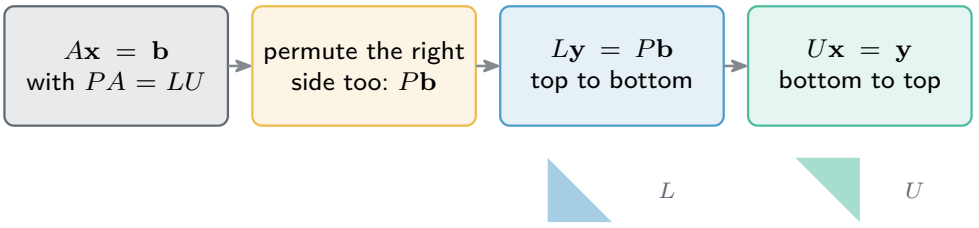
- U is the upper triangular matrix that elimination produces.
- L is lower triangular with 1s on the diagonal; below the diagonal it holds the multipliers used.
- If a zero pivot appears, swap rows: $PA = LU$, where P is a permutation matrix.

Solving with the factors

To solve $Ax = b$ with $PA = LU$:

- Permute the right side: compute Pb.
- Forward substitution: solve $Ly = Pb$ from the top down.
- Back substitution: solve $Ux = y$ from the bottom up.
- $\det A = \pm \prod u_{ii}$, the sign set by the number of row swaps.

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Permute, forward, back.

Rank and nullity of compositions

For linear maps $A : U \to V$ and $B : V \to W$:

- $\ker A \subseteq \ker(BA)$, so $\text{nullity}(BA) \geq \text{nullity}(A)$.
- $\text{nullity}(BA) = \text{nullity}(A) + \dim(\text{im}\, A \cap \ker B) \leq \text{nullity}(A) + \text{nullity}(B)$.
- $\text{rank}(BA) \leq \min(\text{rank}\, A, \text{rank}\, B)$.

Worked example: a 2×2 solve

$L = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$, $b = (5, 12)$.

- $Ly = b$: $y_1 = 5$; $2y_1 + y_2 = 12$, so $y_2 = 2$.
- $Ux = y$: $2x_2 = 2$, so $x_2 = 1$; $3x_1 + 1 = 5$, so $x_1 = \frac{4}{3}$.
- Check: $A = LU = \begin{pmatrix} 3 & 1 \\ 6 & 4 \end{pmatrix}$ sends $(\frac{4}{3}, 1)$ to $(5, 12)$, and $\det A = 3 \times 2 = 6$.

LU solve: forward through L, then backward through U

Stage	Known right side	Result
Ly = b	b = (−1, −1, 12)	y = (−1, 1, 8)
Ux = y	y = (−1, 1, 8)	x = (1, −1, 2)

Solve with LU by forward substitution through L, then back substitution through U

Worked example: nullity bounds

$A : \mathbb{R}^5 \to \mathbb{R}^4$ has rank 3, and $B : \mathbb{R}^4 \to \mathbb{R}^3$ has rank 3.

- nullity $A = 5 - 3 = 2$ and nullity $B = 4 - 3 = 1$.
- $\text{nullity}(BA) = 2 + \dim(\text{im}\, A \cap \ker B)$, which is 2 or 3.
- So $\text{rank}(BA) = 5 - \text{nullity}(BA)$ is 3 or 2. Without more information, both are possible.

Common errors

- Solving $Ly = b$ instead of $Ly = Pb$ when rows were swapped.
- Substituting in the wrong direction through L or U.
- Forgetting the sign from row swaps in the determinant.
- Adding nullities as if equality always held.
- Thinking $\text{rank}(BA)$ can exceed the rank of either factor.

Key terms

forward substitution: Solving a lower triangular system from the first equation downward.

nullity: The dimension of the kernel of a linear map.

Check yourself

- $A = LU$ with U having diagonal 2, 3, 5 and no row swaps. What is $\det A$?
- $A : \mathbb{R}^4 \to \mathbb{R}^4$ has nullity 1, and so does B. What is the largest possible nullity of BA?
- In $PA = LU$, what do you apply P to when solving $Ax = b$?

Answers: 30. 2. To b.

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C.11 · Integration by parts, order reversal and symmetry

GRE Subject Test · GRE Mathematics

Integration by parts, order reversal and symmetry

In 1907 Guido Fubini, in Genoa, proved when the order of a double integral can be swapped. The condition matters. For $f(x,y) = \dfrac{x^2 - y^2}{(x^2+y^2)^2}$ on the unit square, integrating first in y and then in x gives $\pi/4$; the other order gives $-\pi/4$. The function blows up at the corner $(0,0)$, and its absolute value has an infinite integral, so Fubini's theorem does not apply. When it does apply, swapping the order can turn an impossible integral into an easy one.

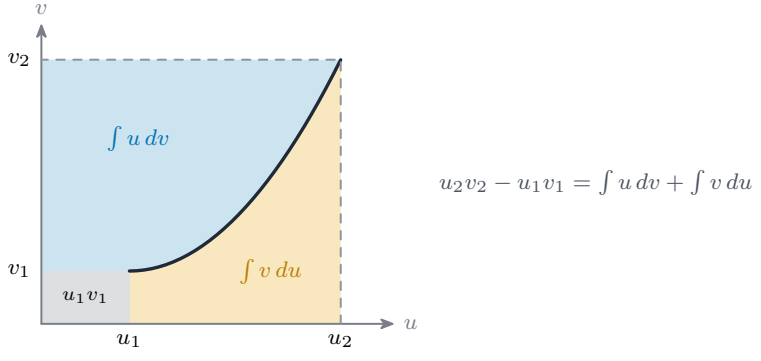
GRE Mathematics integration questions ask you to:

- Use **integration by parts** 分部积分, including repeated and cyclic cases.
- Reverse the order of a double integral by redrawing the region.
- Use **reflection symmetry** 反射对称 and odd–even symmetry to simplify definite integrals.

Integration by parts

$$\int u\,dv = uv - \int v\,du$$

- It comes from the product rule. Choose u to become simpler when differentiated (logs, powers) and dv easy to integrate.
- Repeated parts: $\int x^2 e^x\,dx = (x^2 - 2x + 2)e^x + C$.
- Cyclic case: for $\int e^x \sin x\,dx$, two rounds of parts give back the original integral; solve for it: $\frac{1}{2}e^x(\sin x - \cos x) + C$.



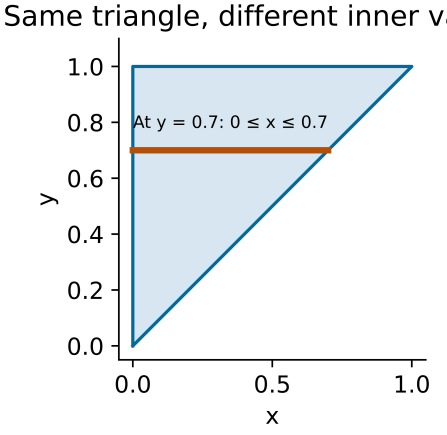
Integration by parts as two areas.

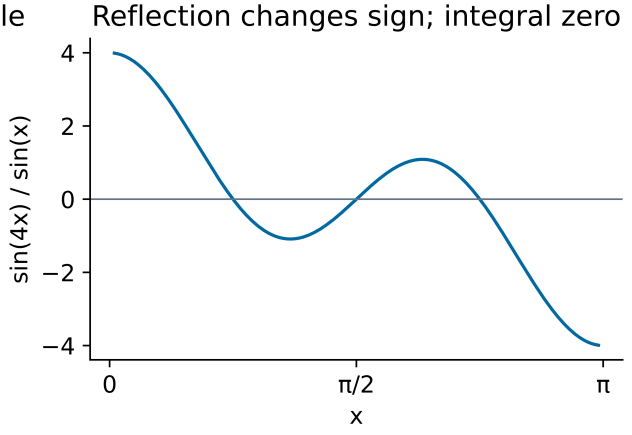
Reversing the order

- Sketch the region described by the limits.
- Read the new limits from the sketch: the outer variable gets constant limits, the inner one runs between curves.
- Fubini: for a continuous function on a bounded closed region (or when $\iint |f|$ is finite), both orders give the same answer.

Symmetry

- If f is odd, $\displaystyle\int_{-a}^{a} f = 0$. If f is even, $\displaystyle\int_{-a}^{a} f = 2\int_{0}^{a} f$.
- Reflection about the midpoint: $\displaystyle\int_{a}^{b} f(x)\,dx = \int_{a}^{b} f(a+b-x)\,dx$.
- If $f(a+b-x) = -f(x)$, the integral is 0. Adding the two forms can also produce an easy integral.





Swap the order of integration by reading the limits from the same region; symmetry can make an integral zero

Worked example: reversing and parts

- $\displaystyle\int_0^1 \int_x^1 e^{y^2}\,dy\,dx$: the region is $0 \le x \le y \le 1$. Reversed, $\displaystyle\int_0^1 \int_0^y e^{y^2}\,dx\,dy = \int_0^1 y e^{y^2}\,dy = \frac{e-1}{2}$.
- $\displaystyle\int_1^x \ln t\,dt = x\ln x - x + 1$, by parts with $u = \ln t$.
- $\displaystyle\int_1^x (\ln t)^2\,dt = x(\ln x)^2 - 2\int_1^x \ln t\,dt = x\left[(\ln x)^2 - 2\ln x + 2\right] - 2$. Check: it is 0 at $x = 1$, and its derivative is $(\ln x)^2$.

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Worked example: symmetry

- $I = \displaystyle\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx$. Replacing x by $\pi/2 - x$ gives $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\,dx$, the same I. Adding: $2I = \pi/2$, so $I = \pi/4$.
- $\displaystyle\int_0^{\pi} \frac{\sin 4x}{\sin x}\,dx$: replacing x by $\pi - x$ changes the sign of the integrand, so the integral is 0.
- $\displaystyle\int_{-1}^{1} x^3 \cos x\,dx = 0$, since the integrand is odd.

Common errors

- Choosing u so that the new integral is harder.
- Dropping the boundary term uv in a definite integral.
- Swapping the limits' numbers without redrawing the region.
- Swapping the order where Fubini's conditions fail.
- Calling a function odd without checking $f(-x) = -f(x)$.

Key terms

integration by parts: The identity $\int u\,dv = uv - \int v\,du$, derived from the product rule.

reflection symmetry: A relation between the values of an integrand at points mirrored across the midpoint of the interval.

Check yourself

- What is $\int xe^x\,dx$?
- Reverse the order: $\displaystyle\int_0^1 \int_y^1 f(x,y)\,dx\,dy$.
- What is $\displaystyle\int_{-2}^{2} x^5 e^{x^2}\,dx$?

Answers: $(x-1)e^x + C$. $\displaystyle\int_0^1 \int_0^x f(x,y)\,dy\,dx$. 0.

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C.12 · Related rates and removable quotient limits

GRE Subject Test · GRE Mathematics

Related rates and removable quotient limits

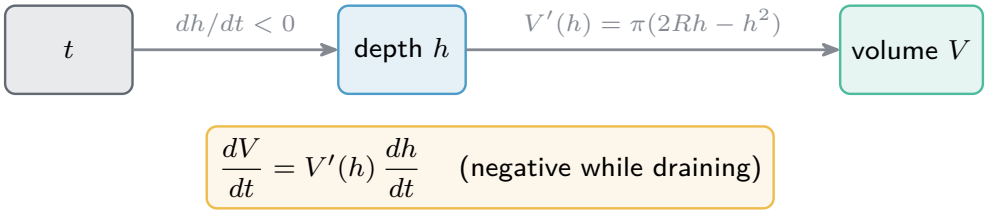
In 1643 Evangelista Torricelli found that water leaves a hole near the bottom of a tank at the speed it would reach falling freely from the water surface: $v = \sqrt{2gh}$. As the tank empties, h falls, so the flow slows, which makes h fall more slowly still. Questions like "how fast is the level dropping when the water is 1 m deep?" link two changing quantities through one equation. Differentiating that equation with respect to time turns it into a **related rate** 相关变化率 problem.

GRE Mathematics questions on rates and limits ask you to:

- Set up and solve related-rate problems with correct signs and units.
- Find a **removable limit** 可去极限 by factoring or rationalising.
- Use standard limits and the squeeze theorem.

Related rates

- Draw the situation and name every changing quantity.
- Write an equation that holds at every moment.
- Differentiate it with respect to time, using the chain rule.
- Only then substitute the values at the moment asked about. Substituting first turns variables into constants.
- A decreasing quantity has a negative rate; report a magnitude only if the question asks for one.



Related rates through the chain rule.

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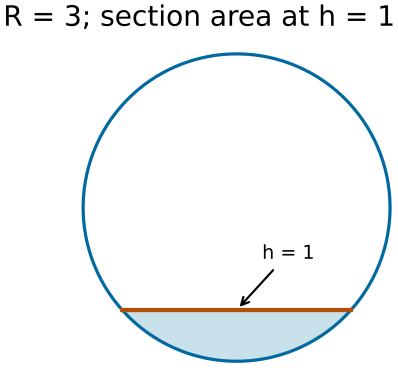
Removable limits

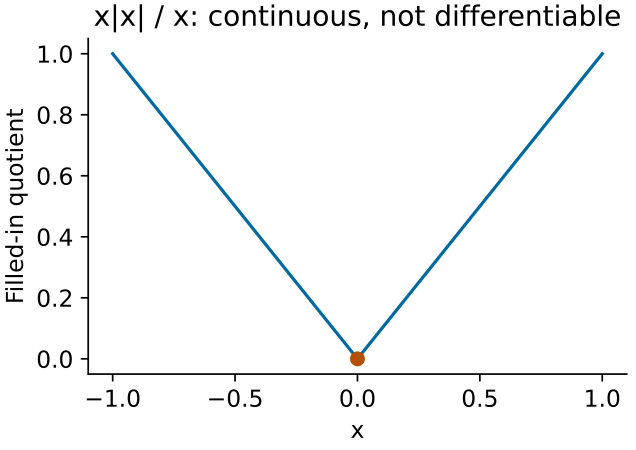
If $f(x)/g(x)$ gives $0/0$ at a, the limit may still exist.

- Factor and cancel: $\dfrac{x^2-9}{x-3} = x+3$ for $x \neq 3$, so the limit is 6.
- Rationalise: $\dfrac{\sqrt{x+4}-2}{x} = \dfrac{1}{\sqrt{x+4}+2} \to \dfrac{1}{4}$.
- Defining $f(a)$ as the limit fills the gap and makes the function continuous there.

Standard limits

- $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$ (with x in radians), so $\lim\limits_{x\to 0} \dfrac{\sin 3x}{x} = 3$.
- $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2} = \dfrac{1}{2}$ and $\lim\limits_{n\to\infty} \left(1+\dfrac{1}{n}\right)^n = e$.
- Squeeze: if $g \le f \le h$ and g, h have the same limit, so does f. $x^2 \sin(1/x) \to 0$ since $|x^2 \sin(1/x)| \le x^2$.





The cross-section at $h = 1$ has radius^2 $= 9 - 2^2 = 5$, so area 5π

Worked example: a spherical tank

A spherical tank of radius $R = 2$ m has water depth h, measured from the bottom. At $h = 1$ m the depth falls at 0.1 m/min.

- A slice at height s has radius squared $2Rs - s^2$, so $V(h) = \pi\left(Rh^2 - \dfrac{h^3}{3}\right)$.
- $\dfrac{dV}{dt} = \pi(2Rh - h^2)\dfrac{dh}{dt} = \pi(4-1)(-0.1) = -0.3\pi$ m^3/min.
- Water leaves at $0.3\pi \approx 0.94$ m^3/min; the minus sign records the loss.

Worked example: a sliding ladder

A 5 m ladder leans on a wall. Its foot slides away at 0.5 m/s. How fast does the top fall when the foot is 3 m from the wall?

- $x^2 + y^2 = 25$, so $x\dfrac{dx}{dt} + y\dfrac{dy}{dt} = 0$.
- At $x = 3$, $y = 4$: $3 \times 0.5 + 4\dfrac{dy}{dt} = 0$, so $\dfrac{dy}{dt} = -0.375$ m/s.
- The top falls at 0.375 m/s; it speeds up as the ladder gets lower.

Common errors

- Substituting values before differentiating.
- Losing the sign of a decreasing quantity.
- Mixing units, such as cm and m, in one rate.
- Using $\sin x/x \to 1$ with x in degrees.
- Concluding a limit does not exist because the function is undefined at the point.

Key terms

related rate: A rate of change found by differentiating an equation that links changing quantities.

removable limit: A finite limit at a point where the function is undefined, which can fill the gap continuously.

Check yourself

- A circle's radius grows at 2 cm/s. How fast is the area growing when $r = 5$ cm?
- What is $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3}$?
- What is $\lim\limits_{x \to 0} \dfrac{\tan x}{x}$?

Answers: $2\pi r \dfrac{dr}{dt} = 20\pi$ cm^2/s. 6. 1.

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T.12 · Metric completeness and closure from a basis

GRE Subject Test · GRE Mathematics

Metric completeness and closure from a basis

In his 1906 doctoral thesis in Paris, Maurice Fréchet asked what the ideas of limit and continuity really need. His answer was a set with a distance function obeying a few rules: what we now call a metric space. Points could be numbers, curves or functions. The real numbers have a property that many such spaces lack: every sequence whose terms crowd together has a limit in the space. Fréchet's framework let mathematicians ask which spaces are complete, and which properties depend on the distance rather than on the open sets.

GRE Mathematics metric and topology questions ask you to:

- Check the metric axioms.
- Decide whether a space is a **complete metric space** 完备度量空间.
- Use pullback metrics and isometries.
- Find closures and interiors from a basis or a **neighbourhood basis** 邻域基.

Metrics

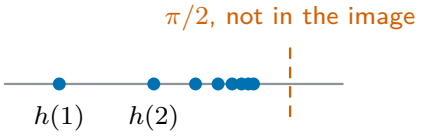
A metric d on a set X satisfies, for all x, y, z:

- $d(x, y) \geq 0$, with $d(x, y) = 0$ exactly when $x = y$.
- Symmetry: $d(x, y) = d(y, x)$.
- Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$.
- $|x - y|^2$ is not a metric on $\mathbb{R}$: $d(0, 2) = 4 > d(0, 1) + d(1, 2) = 2$.

Completeness

- A space is complete when every Cauchy sequence converges to a point of the space.
- $\mathbb{R}$ is complete; $\mathbb{Q}$ and $(0, 1)$ are not. A closed subset of a complete space is complete.
- If f is injective, $d(x, y) = |f(x) - f(y)|$ is a metric, and f is an isometry onto its image. The space is complete exactly when the image is complete.
- Completeness depends on the metric. $\mathbb{R}$ with $|\arctan x - \arctan y|$ has the usual open sets but is not complete.

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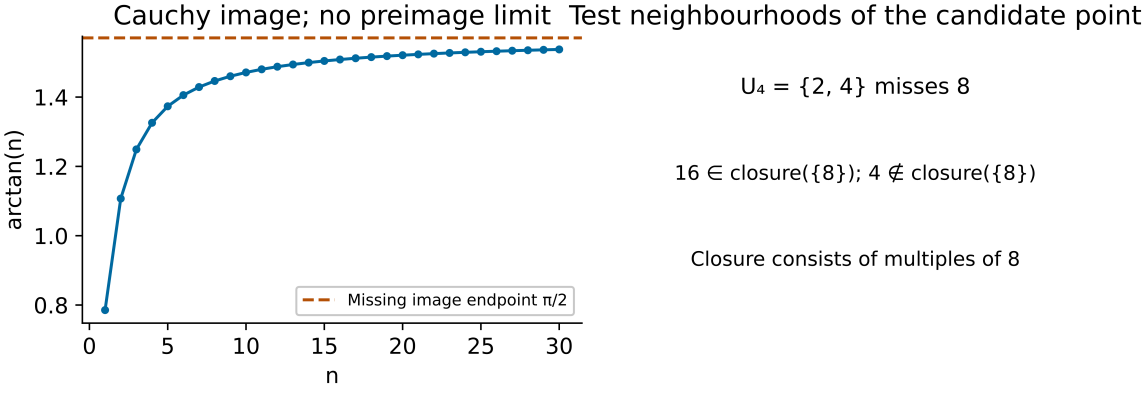
$d(m, n) = |\arctan m - \arctan n|$: the integers become Cauchy but converge to nothing, because the image is not complete

A pullback metric is complete only if the image is.

Bases and closures

A basis for a topology is a collection of open sets such that every open set is a union of them.

- A point x is in the closure of A exactly when every basic open set containing x meets A.
- A point is in the interior of A when some basic open set containing it lies inside A.
- Changing the basis can change the closure, even for the same set.



The image of a Cauchy sequence can converge to a point outside the image

Worked example: two pullback metrics

On $\mathbb{R}$, let $d_1(x, y) = |x^3 - y^3|$ and $d_2(x, y) = |\arctan x - \arctan y|$.

- Both are metrics: x^3 and $\arctan x$ are injective, and absolute value gives symmetry and the triangle inequality.
- x^3 maps $\mathbb{R}$ onto $\mathbb{R}$, which is complete, so d_1 is complete.
- $\arctan$ maps onto $(-\pi/2, \pi/2)$. The sequence $x_n = n$ is d_2-Cauchy, since $\arctan n \to \pi/2$, but no real number has image $\pi/2$. So d_2 is not complete.

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Worked example: a different basis

The lower-limit topology on $\mathbb{R}$ has the basis of half-open intervals $[a, b)$. Find the closure of $A = (0, 1)$.

- The point 0: every basic set $[0, \varepsilon)$ meets A, so 0 is in the closure.
- The point 1: the basic set $[1, 1.5)$ does not meet A, so 1 is not in the closure.
- The closure is $[0, 1)$, while in the usual topology it is $[0, 1]$.

Common errors

- Assuming a bounded metric space is complete.
- Treating completeness as a property of the open sets alone.
- Forgetting to check the triangle inequality.
- Using the usual closure when a different basis is given.
- Thinking a closed set in an incomplete space must be complete.

Key terms

complete metric space: A metric space in which every Cauchy sequence converges to a point of the space.

neighbourhood basis: A collection of open sets around a point that is enough to test closeness to that point.

Check yourself

- Is $\mathbb{Q}$ with the usual distance complete?
- Is $[0, \infty)$ with the usual distance complete?
- In the lower-limit topology, is $[0, 1)$ open?

Answers: no. Yes: it is closed in $\mathbb{R}$. Yes: it is a basic open set.

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