

T.12 · Metric completeness and closure from a basis

GRE · GRE Subject Test · GRE Mathematics

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Metric axioms, Cauchy sequences and topology from a basis.

- Check a pullback metric using the properties of its defining map
- Decide completeness through the image of an isometry
- Determine closure using every basic neighbourhood instead of Euclidean intuition

complete metric space : A metric space in which every Cauchy sequence converges to a point of that space.

neighbourhood basis : Basic neighbourhoods sufficient to test local topological properties.

Choose and justify a method

For an injective map $h:X\rightarrow\mathbb{R}$, $d(x,y)=|h(x)-h(y)|$ is a metric: symmetry and the triangle inequality come from $\mathbb{R}$, and injectivity ensures $d(x,y)=0$ only when $x=y$. The map h is an isometry onto its image $h(X)$. If h is not injective, this construction may be only a pseudometric. Bounded distances do not prove a space complete, and a familiar set of points can have very different Cauchy behaviour under different metrics.

The pullback metric is complete exactly when the image $h(X)$ is complete in the ordinary real distance. A closed subset of $\mathbb{R}$ is complete. For $h(x)=\arctan x$ on $\mathbb{R}$, the image is $(-\pi/2,\pi/2)$, which omits its endpoints. The sequence n is Cauchy in the pullback metric because $\arctan n\rightarrow\pi/2$, but no real x has $\arctan x=\pi/2$, so the metric is incomplete. For $h(x)=x^3$, the image is all $\mathbb{R}$ and the pullback metric is complete. Topological equivalence alone does not preserve completeness.

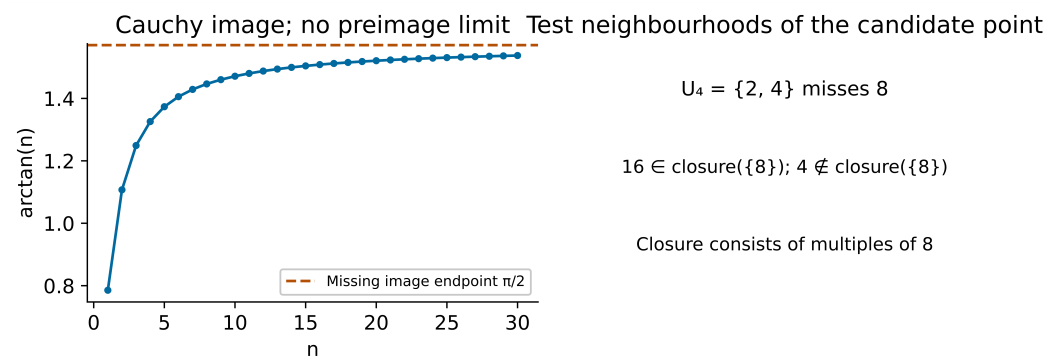
In a topology with a basis, x lies in the closure of A when every basic open neighbourhood of x meets A . This is a membership test, not automatically a Euclidean endpoint operation. The basis must cover the space, and for a point in two basis sets there must be a smaller basis set containing it within their intersection. A point can lie in the closure of a singleton even when it is not the singleton's own point; that possibility depends on the topology.

On $X=\{2,3,4,\dots\}$, take basic sets $U_k=\{n \text{ in } X: n \text{ divides } k\}$, for $k\geq 2$. They cover X since $x\in U_x$; intersections are $U_{\gcd(k,l)}$ when the gcd is at least 2, or empty. A point x lies in closure of $\{n\}$ exactly when every k divisible by x is also divisible by n : equivalently n divides x . More directly, the smallest basic neighbourhood U_x

contains n precisely when n divides x , and every other neighbourhood containing x includes these divisors. Thus closure of $\{8\}$ consists of positive multiples of 8, not the divisors of 8.

Worked reasoning

Under $d(x,y)=|\arctan x-\arctan y|$, the integer sequence goes toward a missing image endpoint rather than a point of $\mathbb{R}$, so it is Cauchy without convergence in that space. In the divisor basis, 16 lies in closure of $\{8\}$, since any basic set containing 16 also contains 8. But 4 does not: $U_4=\{2,4\}$ is a neighbourhood of 4 missing 8. This explicitly separates multiples from divisors.



Original course illustration; its values belong to the worked example, not the later practice.

Conditions and counterexamples

A Cauchy limit must belong to the same space. Completeness is a metric property, not just a topological one. For closure, test every neighbourhood of the candidate point, rather than every neighbourhood of the singleton value.

Guided application

On $\mathbb{R}$ compare $d_1(x,y)=|x^3-y^3|$ and $d_2(x,y)=|\arctan x-\arctan y|$. Prove they are metrics and decide completeness.

■ Worked solution

Both defining maps are injective, giving positive separation; absolute distance gives symmetry and the triangle inequality. Each map is an isometry onto its real image. The cube map has image all of $\mathbb{R}$, a complete space, so d_1 is complete. The arctangent image is the open interval $(-\pi/2,\pi/2)$, which is incomplete. The sequence n is d_2 -Cauchy because $\arctan n$ tends to $\pi/2$, but no real point has that image, so it has no d_2 -limit. Bounded metric values do not imply completeness.

Independent transfer

On $X = \{2, 3, 4, \dots\}$ use the divisor basis $U_k = \{n \in X : n \mid k\}$. Find the closure of $\{6\}$ and the interior of $\{6\}$. Verify the basis conditions, including empty intersections.

■ Check after attempting

Every x belongs to U_x . An intersection is $U_{\gcd(k,l)}$ if the gcd is at least two, and is empty for gcd one. Thus a point in an intersection has a basic neighbourhood within it. The smallest basic neighbourhood of x is U_x , contained in every U_k that contains x . It meets $\{6\}$ exactly when $6 \mid x$. Therefore the closure is $\{6, 12, 18, \dots\}$, the multiples of six. The point six is not interior to its singleton because U_6 also contains two and three. No other point is in that singleton, so its interior is empty. This closure calculation uses all basic neighbourhoods, not Euclidean intuition.