

C.11 • Integration by parts, order reversal and symmetry

GRE • GRE Subject Test • GRE Mathematics

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Product rule, improper limits and planar integration regions.

- Derive integration-by-parts reductions with valid endpoint limits
- Reverse a double integral by reconstructing its region
- Use reflection symmetry while checking removable endpoint behaviour

integration by parts : An integration identity derived from the product rule with a boundary term.

reflection symmetry : A relation between integrand values at points mirrored across an interval midpoint.

Choose and justify a method

Integration by parts comes from the product rule: $\int u \, dv = uv - \int v \, du$. For definite integrals include the boundary term at both ends. With $I_n(x) = \int_1^x (\ln t)^n \, dt$, choose $u = (\ln t)^n$ and $dv = dt$. For $n \geq 1$ the lower boundary vanishes, giving $I_n = x(\ln x)^n - n I_{n-1}$. A plus sign would violate the differentiated identity. Keep the same lower limit in a recurrence; changing it changes the constants.

For improper integrals first apply the identity on finite endpoints, then justify the limiting boundary and remaining integral. If $\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}$, set $u = x$ and $dv = x e^{-x^2} \, dx$ for $\int x^2 e^{-x^2} \, dx$. Since $v = -e^{-x^2}/2$ and $x e^{-x^2} \rightarrow 0$ at both infinities, the full second moment is $\sqrt{\pi}/2$. Evenness gives the positive-half-line moment $\sqrt{\pi}/4$. The vanished boundary is part of the argument, not an automatic property of every improper integral.

To reverse $\int_0^1 \int_x^1 F(x,y) \, dy \, dx$, describe the triangle $0 \leq x \leq y \leq 1$, then rewrite it as $\int_0^1 \int_0^y F(x,y) \, dx \, dy$. Both the outer interval and inner bounds change; swapping symbols alone changes the region. For $F = e^{-(y^2)}$, integrating over x first gives $\int_0^1 y e^{-(y^2)} \, dy = (e-1)/2$. Continuity on this compact triangle makes order reversal valid; singular or conditionally convergent cases require stronger care.

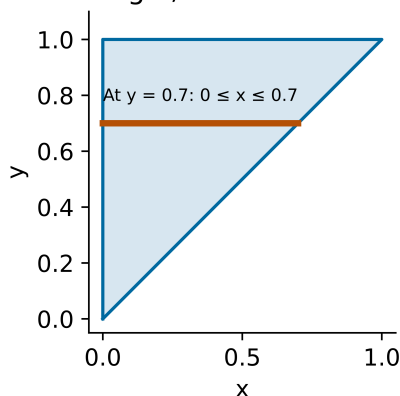
If an integrable function on $[a,b]$ satisfies $f(a+b-x) = -f(x)$, reflection makes its integral equal to its negative, hence zero. For $\sin(2mx)/\sin x$ on $[0,\pi]$ with positive integer m , reflection $x \mapsto \pi - x$ changes the numerator sign and preserves the denominator. The apparent endpoint singularities are removable: limits are $2m$ at zero and $-2m$ at π . Check these limits before invoking symmetry; cancellation is not a substitute for

integrability.

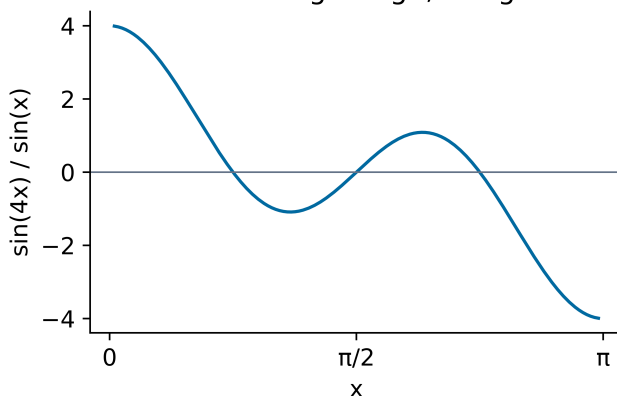
Worked reasoning

For $I_2(x) = \int_1^x (\ln t)^2 dt$, the recurrence gives $x(\ln x)^2 - 2I_1(x)$, with $I_1(x) = x \ln x - x + 1$. Therefore $I_2 = x[(\ln x)^2 - 2\ln x + 2] - 2$. The constant -2 ensures $I_2(1) = 0$. Separately, reversing the triangular $e^{\widehat{(y^2)}}$ integral produces a factor y from the inner x -length; dropping that factor would restore the original difficulty and change the value.

Same triangle, different inner variable



Reflection changes sign; integral zero



Original course illustration; its values belong to the worked example, not the later practice.

Conditions and counterexamples

Improper integration by parts needs endpoint limits. Reconstruct the region before reversing order. An odd-looking reflected integrand must be integrable, including any removable endpoints.

Guided application

Evaluate $\int_0^1 \int_x^1 e^{y^2} dy dx$ by reversing order. Describe the region before and after reversal. Derive a formula for $I_2(x) = \int_1^x (\ln t)^2 dt$, $x > 0$.

■ Worked solution

The region is $0 \leq x \leq y \leq 1$. In the reversed order, y ranges from zero to one and x from zero to y . Continuity on the compact triangle permits order reversal, giving $\int_0^1 y e^{y^2} dy = (e - 1)/2$. Integration by parts with $u = (\ln t)^2$, $dv = dt$ gives $I_2 = x(\ln x)^2 - 2I_1$. Since $I_1 = x \ln x - x + 1$,

$$I_2 = x[(\ln x)^2 - 2\ln x + 2] - 2.$$

Differentiation checks the integrand; substituting $x=1$ checks the constant.

Independent transfer

For positive integer m , evaluate $\int_0^\pi \sin(2mx)/\sin x \, dx$. Justify endpoint integrability before invoking symmetry.

■ Check after attempting

The endpoint limits are $2m$ at zero and $-2m$ at π , by the first-order sine limit. Both singularities are removable, so the continuously extended integrand is integrable. Reflection $x \mapsto \pi - x$ preserves the denominator and reverses the numerator. The integral equals its negative and therefore is zero. Symmetric cancellation would not establish convergence for a genuinely divergent endpoint; the removable limits are essential.