

A.7 · LU factorisation and nullity of composed maps

GRE · GRE Subject Test · GRE Mathematics

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Row operations, triangular systems and rank-nullity.

- Solve a factored linear system by forward and backward substitution
- Apply row permutations consistently when pivoting is required
- Bound the kernel dimension of a composition using image-kernel intersections

forward substitution : Solving a lower triangular system from its first equation downward.

nullity : The dimension of the kernel of a linear map.

Choose and justify a method

A factorisation $A=LU$ expresses a square matrix as lower triangular L and upper triangular U . With unit diagonal L , forward substitution is especially simple. An invertible matrix need not admit this form without row exchanges: a zero leading pivot can require a permutation. For $PA=LU$, solve $Ly=Pb$, then $Ux=y$. The permutation acts on the right side as well as the coefficient matrix. Nonzero leading principal pivots justify the usual no-exchange elimination, not invertibility alone.

In $Ly=b$, compute y from top to bottom, subtracting terms already known; divide by the current diagonal unless it is one. In $Ux=y$, compute x from bottom to top. For each equation substitute back into the original row as a check. Reusing LU for many right sides avoids repeating the elimination: dense factorisation takes cubic-order work in dimension, while each triangular solve takes quadratic-order work. Exact arithmetic can check small examples; numerical pivoting reduces some roundoff problems but cannot remove inherent ill-conditioning.

For $B:V \rightarrow W$ and $A:W \rightarrow Z$, $\ker B$ is contained in $\ker(A \circ B)$, but the latter can be larger. Extra vectors are those mapped by B into $\ker A$. Restrict B to $\ker(A \circ B)$: its image is $\text{im } B \cap \ker A$ and its kernel is $\ker B$. Rank-nullity on this restricted map gives $\dim \ker(A \circ B) = \dim \ker B + \dim(\text{im } B \cap \ker A)$. The intersection, not the whole kernel of A , determines the extra nullity.

For endomorphisms of $\mathbb{R}^6$ with nullity $A=2$ and nullity $B=3$, rank $B=3$ and the intersection dimension can range from 0 to 2. Thus nullity of $A \circ B$ ranges from 3 to 5. Bounds depend on the common intermediate space: for subspaces of dimensions r and s in an m -dimensional space, their intersection has dimension between $\max(0, r+s-m)$

and $\min(r,s)$. Reversing the composition can change its nullity, even though AB and BA are both defined. Choose compatible spaces before applying these formulas.

Worked reasoning

Let $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, and $b = (-1, -1, 12)$. Forward substitution gives $y = (-1, 1, 8)$. Back substitution gives $x_3 = 2$, $x_2 = (1 - 4)/3 = -1$, and $x_1 = (-1 - (-1) + 2)/2 = 1$. Therefore $x = (1, -1, 2)$. Multiplying Ux gives y and multiplying Ly gives b , independently checking the factor order.

LU solve: forward through L , then backward through U

Stage	Known right side	Result
$Ly = b$	$b = (-1, -1, 12)$	$y = (-1, 1, 8)$
$Ux = y$	$y = (-1, 1, 8)$	$x = (1, -1, 2)$

Original course illustration; its values belong to the worked example, not the later practice.

Conditions and counterexamples

Solve with L first and U second, and permute b if $PA=LU$. A composition’s nullity is not automatically the sum of the two nullities; only the image-kernel intersection adds to nullity B .

Guided application

Let $L = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ and $b = (5, 12)^T$. Solve $LUx=b$ and verify in the multiplied matrix. For $PA=LU$, where does P act when solving $Ax=b$?

■ Worked solution

First solve $Ly=b$: $y_1 = 5$, $2y_1 + y_2 = 12$, so $y_2 = 2$. Then $Ux=y$ gives $x_2 = 1$ and $3x_1 + 1 = 5$, hence $x_1 = 4/3$. The product matrix is $A = \begin{pmatrix} 3 & 1 \\ 6 & 4 \end{pmatrix}$; applying it to $(4/3, 1)^T$ gives $(5, 12)^T$. If $PA=LU$, multiply the original equation by P and solve $Ly=Pb$, followed by $Ux=y$. Omitting P on b solves a different system.

Independent transfer

Let A and B be endomorphisms of $\mathbb{R}^5$ with nullities two and three, respectively. Find all possible nullities of AB , and justify both the bound and attainability.

■ Check after attempting

Rank B is two. Restrict B to $\ker(AB)$ to get

$$\dim \ker(AB) = \dim \ker B + \dim(\operatorname{im} B \cap \ker A).$$

The two subspaces in the intersection both have dimension two in five dimensions, so intersection dimension may be zero, one or two. Thus nullity is 3, 4 or 5. To realise each, take $\ker A = \operatorname{span}(e_1, e_2)$ and image B respectively $\operatorname{span}(e_3, e_4)$, $\operatorname{span}(e_1, e_3)$ or $\operatorname{span}(e_1, e_2)$. Choose any rank-two B onto that image. These constructions prove attainability rather than only a loose interval bound.