

## C.10 · Implicit differentiation and the inverse Jacobian

GRE · GRE Subject Test · GRE Mathematics

### Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Chain rule, matrix inversion and local inverse hypotheses.

- Differentiate a coupled implicit system as a linear system
- Use a nonzero Jacobian determinant to justify a local inverse
- Distinguish inverse partial derivatives from scalar reciprocal rules

**Jacobian matrix** : The matrix of first partial derivatives of a vector-valued map.

**local inverse** : An inverse defined on neighbourhoods of a particular input and output point.

### Choose and justify a method

For a differentiable map  $F(u,v)=(x,y)=(f(u,v),g(u,v))$ , its Jacobian is  $J=[f_u, f_v, g_u, g_v]$ . Small changes satisfy  $[dx, dy]$  (transpose)  $= J[du, dv]$  (transpose) to first order. To find  $u_x$  while holding  $y$  fixed, differentiate both defining equations with respect to  $x$ :  $f_u u_x + f_v v_x = 1$  and  $g_u u_x + g_v v_x = 0$ . These are a coupled linear system, not two independent scalar inverse rules.

If  $f$  and  $g$  are continuously differentiable near the point and  $\det J = f_u g_v - f_v g_u$  is nonzero there, the inverse-function theorem supplies a differentiable local inverse. Its derivative is  $J^{-1} = (1/\det J)[[g_v, -f_v], [-g_u, f_u]]$ . Thus  $u_x = g_v/\det J$ ,  $u_y = -f_v/\det J$ ,  $v_x = -g_u/\det J$  and  $v_y = f_u/\det J$ . Evaluate every derivative at the corresponding point. A local inverse need not extend to a global one.

For implicit equations  $H(u,v,x,y)=0$  and  $K(u,v,x,y)=0$ , differentiate while fixing the requested independent coordinate. Solve  $[[H_u, H_v], [K_u, K_v]][u_x, v_x]$  (transpose)  $= -[H_x, K_x]$  (transpose). The determinant in the unknown variables  $u, v$  must be nonzero to use the usual implicit-function theorem. Signs on the right come from moving known derivatives to the other side. If the determinant vanishes, this theorem is inconclusive; it does not by itself prove no inverse or no implicit solution exists.

The scalar shortcut  $du/dx = 1/(dx/du)$  holds for a one-variable inverse with nonzero derivative, but usually fails for a coupled system because  $v$  changes to keep  $y$  fixed. For  $x=u+v, y=u+2v$ ,  $J = [[1, 1], [1, 2]]$  has determinant 1 and inverse  $[[2, -1], [-1, 1]]$ . Therefore  $u_x = 2$  while  $1/f_u = 1$ . An inverse Jacobian transforms differential sensitivities; a change-of-variables integral instead uses the absolute determinant for area or volume

scaling, not a selected inverse entry.

### Worked reasoning

Take  $x=u^2+v$  and  $y=u-v$ . At  $(u,v)=(1,0)$ , the output is  $(1,1)$ , and  $J = [[2, 1], [1, -1]]$  has determinant  $-3$ . The inverse is  $[[1/3, 1/3], [1/3, -2/3]]$ , so  $u_x = 1/3$  and  $v_y = -2/3$  there. Multiplying  $J$  by this inverse gives the identity. The reciprocal  $1/f_u = 1/2$  is not  $u_x$  because changing  $u$  also requires changing  $v$  to keep  $y$  fixed.

### Coupled sensitivity: inverse entries differ from reciprocals

| Forward map                                      | Jacobian            | Inverse Jacobian            |
|--------------------------------------------------|---------------------|-----------------------------|
| $x = u + v; y = u + 2v$                          | $[[1, 1], [1, 2]]$  | $[[2, -1], [-1, 1]]$        |
| $x = u^2 + v; y = u - v$<br>at $(u, v) = (1, 0)$ | $[[2, 1], [1, -1]]$ | $[[1/3, 1/3], [1/3, -2/3]]$ |

*Original course illustration; its values belong to the worked example, not the later practice.*

### Conditions and counterexamples

Differentiate both equations and state which output is held fixed. Nonzero determinant guarantees a local inverse under the regularity hypotheses; zero determinant is not a proof of impossibility. Use an inverse entry for sensitivities and an absolute determinant for integration.

### Guided application

Let  $x = u^2 + v, y = u - v$ . At  $(u, v) = (1, 0)$ , find the inverse Jacobian and the four inverse partial derivatives. Check by multiplication.

### ■ Worked solution

The corresponding output is  $(1, 1)$ . The Jacobian is  $J = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$  with determinant  $-3$ , nonzero. Continuous partials give a differentiable local inverse. Thus

$$J^{-1} = \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{pmatrix}.$$

Read rows as  $(u_x, u_y)$  and  $(v_x, v_y)$ , all at output  $(1, 1)$ . Multiplication gives the identity. In particular  $u_x = 1/3$ , not  $1/x_u = 1/2$ :  $v$  changes to keep  $y$  fixed.

### Independent transfer

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For  $F(u, v) = (u^2 - v^2, 2uv)$ , decide whether its derivative is invertible away from the origin and whether  $F$  is globally injective on the punctured plane. What does a zero determinant at the origin establish by itself?

### ■ Check after attempting

$\det DF = 4(u^2 + v^2) > 0$  away from zero, so the inverse-function theorem supplies a local inverse at each such point. But  $F(u, v) = F(-u, -v)$ , so it is not globally injective on the punctured plane. A vanishing determinant at zero means this theorem cannot guarantee a differentiable local inverse there. It is not, by itself, a general proof that no local set-theoretic inverse can exist for any map; for example the scalar cube map is bijective despite derivative zero at zero.