

C.10 · Implicit differentiation and the inverse Jacobian

GRE · GRE Subject Test · GRE Mathematics

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Chain rule, matrix inversion and local inverse hypotheses.

- Differentiate a coupled implicit system as a linear system
- Use a nonzero Jacobian determinant to justify a local inverse
- Distinguish inverse partial derivatives from scalar reciprocal rules

Jacobian matrix : The matrix of first partial derivatives of a vector-valued map.

local inverse : An inverse defined on neighbourhoods of a particular input and output point.

Choose and justify a method

For a differentiable map $F(u,v)=(x,y)=(f(u,v),g(u,v))$, its Jacobian is $J=[f_u, f_v, g_u, g_v]$. Small changes satisfy $[dx, dy]$ (transpose) $= J[du, dv]$ (transpose) to first order. To find u_x while holding y fixed, differentiate both defining equations with respect to x : $f_u u_x + f_v v_x = 1$ and $g_u u_x + g_v v_x = 0$. These are a coupled linear system, not two independent scalar inverse rules.

If f and g are continuously differentiable near the point and $\det J = f_u g_v - f_v g_u$ is nonzero there, the inverse-function theorem supplies a differentiable local inverse. Its derivative is $J^{-1} = (1/\det J)[[g_v, -f_v], [-g_u, f_u]]$. Thus $u_x = g_v/\det J$, $u_y = -f_v/\det J$, $v_x = -g_u/\det J$ and $v_y = f_u/\det J$. Evaluate every derivative at the corresponding point. A local inverse need not extend to a global one.

For implicit equations $H(u,v,x,y)=0$ and $K(u,v,x,y)=0$, differentiate while fixing the requested independent coordinate. Solve $[[H_u, H_v], [K_u, K_v]][u_x, v_x]$ (transpose) $= -[H_x, K_x]$ (transpose). The determinant in the unknown variables u, v must be nonzero to use the usual implicit-function theorem. Signs on the right come from moving known derivatives to the other side. If the determinant vanishes, this theorem is inconclusive; it does not by itself prove no inverse or no implicit solution exists.

The scalar shortcut $du/dx = 1/(dx/du)$ holds for a one-variable inverse with nonzero derivative, but usually fails for a coupled system because v changes to keep y fixed. For $x=u+v, y=u+2v$, $J = [[1, 1], [1, 2]]$ has determinant 1 and inverse $[[2, -1], [-1, 1]]$. Therefore $u_x = 2$ while $1/f_u = 1$. An inverse Jacobian transforms differential sensitivities; a change-of-variables integral instead uses the absolute determinant for area or volume

scaling, not a selected inverse entry.

Worked reasoning

Take $x=u^2+v$ and $y=u-v$. At $(u,v)=(1,0)$, the output is $(1,1)$, and $J=[[2,1],[1,-1]]$ has determinant -3 . The inverse is $[[1/3,1/3],[1/3,-2/3]]$, so $u_x=1/3$ and $v_y=-2/3$ there. Multiplying J by this inverse gives the identity. The reciprocal $1/f_u=1/2$ is not u_x because changing u also requires changing v to keep y fixed.

Coupled sensitivity: inverse entries differ from reciprocals

Forward map	Jacobian	Inverse Jacobian
$x = u + v; y = u + 2v$	$[[1, 1], [1, 2]]$	$[[2, -1], [-1, 1]]$
$x = u^2 + v; y = u - v$ at $(u, v) = (1, 0)$	$[[2, 1], [1, -1]]$	$[[1/3, 1/3], [1/3, -2/3]]$

Original course illustration; its values belong to the worked example, not the later practice.

Conditions and counterexamples

Differentiate both equations and state which output is held fixed. Nonzero determinant guarantees a local inverse under the regularity hypotheses; zero determinant is not a proof of impossibility. Use an inverse entry for sensitivities and an absolute determinant for integration.

Guided application

Let $x = u^2 + v, y = u - v$. At $(u, v) = (1, 0)$, find the inverse Jacobian and the four inverse partial derivatives. Check by multiplication.

■ Worked solution

The corresponding output is $(1, 1)$. The Jacobian is $J = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$ with determinant -3 , nonzero. Continuous partials give a differentiable local inverse. Thus

$$J^{-1} = \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{pmatrix}.$$

Read rows as (u_x, u_y) and (v_x, v_y) , all at output $(1, 1)$. Multiplication gives the identity. In particular $u_x = 1/3$, not $1/x_u = 1/2$: v changes to keep y fixed.

Independent transfer

For $F(u, v) = (u^2 - v^2, 2uv)$, decide whether its derivative is invertible away from the origin and whether F is globally injective on the punctured plane. What does a zero determinant at the origin establish by itself?

■ Check after attempting

$\det DF = 4(u^2 + v^2) > 0$ away from zero, so the inverse-function theorem supplies a local inverse at each such point. But $F(u, v) = F(-u, -v)$, so it is not globally injective on the punctured plane. A vanishing determinant at zero means this theorem cannot guarantee a differentiable local inverse there. It is not, by itself, a general proof that no local set-theoretic inverse can exist for any map; for example the scalar cube map is bijective despite derivative zero at zero.