

C.9 · Integrating factors and nonhomogeneous differential equations

GRE · GRE Subject Test · GRE Mathematics

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: First-order product rule, exponentials and characteristic equations.

- Solve first-order linear equations using an integrating factor
- Construct the homogeneous and particular parts of a constant-coefficient solution
- Handle resonance and verify the result in the original equation

integrating factor : A multiplier that turns a first-order linear equation into a product derivative.

particular solution : One solution supplying the specified nonhomogeneous forcing.

Choose and justify a method

A first-order linear equation has the form $y' + p(x)y = q(x)$ on an interval where its coefficients are continuous. It need not be separable. Set $\mu(x) = \exp(\int p(x)dx)$. Since $\mu' = p\mu$, multiplying gives $(\mu y)' = \mu q$, hence $y = \mu^{-1}(\int \mu q dx + C)$. Choose any convenient antiderivative for p ; its integration constant only rescales μ and cancels from the solution. Apply initial data after integration, and keep any singular coefficient points outside the chosen interval.

For $y' + 2xy = x$ with $y(0) = 1$, $\mu = e^{x^2}$. Then $(e^{x^2}y)' = x e^{x^2}$, whose integral is $e^{x^2}/2 + C$. Thus $y = 1/2 + C e^{-x^2}$, and the initial condition gives $C = 1/2$. Direct substitution checks $y' + 2xy = x$. The integrating factor makes the left side a product derivative; it is not an extra factor that remains multiplying the original right side in the final answer.

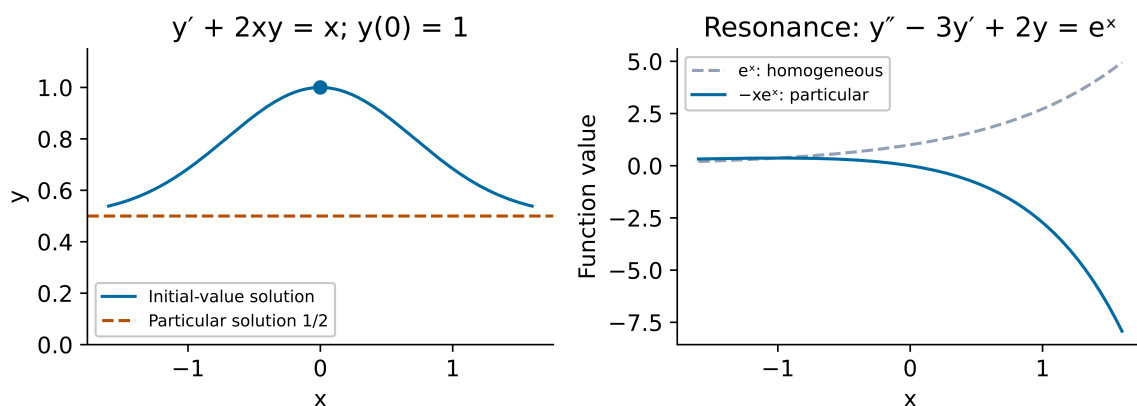
For $ay'' + by' + cy = r(x)$ with a nonzero, solve the characteristic equation $a\lambda^2 + b\lambda + c = 0$ for the homogeneous part. Distinct real roots give two exponentials; a repeated root λ gives $(C_1 + C_2x)e^{\lambda x}$; roots $\alpha \pm i\beta$ give $e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$. The complete solution is $y_h + y_p$. A polynomial forcing suggests a polynomial particular trial; an exponential or sine/cosine forcing suggests the corresponding family, with enough coefficients to account for differentiation.

If a particular trial duplicates a homogeneous solution, it cannot produce the forcing: multiply by x once for a simple root, twice for a repeated root. For $y'' - 3y' + 2y = e^x$, the naive Ke^x is annihilated. Trying Kxe^x gives $-Ke^x$, so $y_p = -xe^x$. Initial conditions determine the homogeneous constants only after adding y_p . Substitute

the final function into the original equation to check signs and forcing; an initial-value check alone cannot verify the differential equation.

Worked reasoning

For $y'' - 3y' + 2y = 4$, the characteristic roots are 1 and 2. A constant particular solution $y_p = 2$ gives $2y_p = 4$, so $y = C_1 e^x + C_2 e^{2x} + 2$. For $y' + 2xy = x$ and $y(0) = 1$, the separate first-order solution is $y = (1 + e^{-x^2})/2$. These illustrate distinct methods; neither equation becomes homogeneous just because its left side is linear.



Original course illustration; its values belong to the worked example, not the later practice.

Conditions and counterexamples

Keep the particular solution. A resonant trial needs an x factor. The integrating-factor derivation applies on a valid coefficient interval and should not divide by y or silently discard zero solutions.

Guided application

Solve $y' + 2y = e^x$, $y(0) = 1$, by an integrating factor. Check the answer directly.

■ Worked solution

The coefficients are continuous everywhere. The integrating factor is $\mu = e^{2x}$. Multiplication gives $(e^{2x}y)' = e^{3x}$. Integration gives $y = e^x/3 + Ce^{-2x}$. The initial condition yields $C = 2/3$, so $y = e^x/3 + 2e^{-2x}/3$. Differentiation gives $y' = e^x/3 - 4e^{-2x}/3$; adding $2y$ gives e^x , and $y(0) = 1$.

Independent transfer

Solve $y'' - 2y' + y = e^x$ with $y(0) = 0, y'(0) = 0$. Explain why neither Ke^x nor Kxe^x is a sufficient particular trial.

■ Check after attempting

The homogeneous root is 1 with multiplicity two, so $y_h = (C_1 + C_2x)e^x$. Both suggested trials already belong to that homogeneous space and are annihilated. Put $y = e^xv$; then $y'' - 2y' + y = e^xv''$. Therefore $v'' = 1$ and $v = x^2/2 + C_2x + C_1$. The two initial conditions give $C_1 = C_2 = 0$, hence $y = x^2e^x/2$. This substitution also checks the forcing without relying only on initial values.