

T.10 • Vector geometry, projections and oriented area

GRE • GRE Subject Test • GRE Mathematics

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Coordinate vectors, dot/cross products and norms.

- Use dot products to classify angles and compute projections
- Calculate triangle area and orientation using a cross product
- Construct or rule out planar dot-product sign configurations

orthogonal : Having a zero dot product in a real inner-product space.

cross product : An oriented perpendicular vector in three dimensions whose magnitude is spanned parallelogram area.

Choose and justify a method

For nonzero real vectors u, v , $u \cdot v = |u||v| \cos \theta$. A positive, zero or negative dot product corresponds to an acute, right or obtuse smaller angle. The scalar projection of v along u is $(v \cdot u)/|u|$; its vector projection is $((v \cdot u)/(u \cdot u))u$. The difference from this projection is orthogonal to u . Do not confuse a projected vector with its signed scalar component. The zero vector is orthogonal to every vector but has no defined angle direction.

For three-dimensional vectors, $u \times v$ is perpendicular to both with length $|u||v| \sin \theta$. Coordinate calculation uses $(u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$. This length is the parallelogram area, so a triangle from two edge vectors has half that area. Reversing their order reverses the cross product but preserves area. Build both edge vectors from the same vertex; crossing two unrelated position vectors generally measures the wrong triangle.

The plane through a point p with nonzero normal n has equation $n \cdot (x - p) = 0$. Distance from q to the plane is $|n \cdot (q - p)|/|n|$; the denominator normalises the scale of the equation. A scalar triple product $u \cdot (v \times w)$ gives signed parallelepiped volume; its absolute value is geometric volume. Zero triple product means dependence of the three edge vectors, not necessarily that each pair is perpendicular or parallel.

For four planar vectors, there are six unordered dot products. The configuration $e_1, -e_1, e_2, -e_2$ has two negative products and four zeros; e_1, e_1, e_2, e_2 has two positive products and four zeros. Four nonzero vectors cannot have every pairwise dot product negative. Order their directions around the circle: each consecutive angular gap would have to exceed 90° , forcing the sum of four gaps above 360° . A zero vector cannot

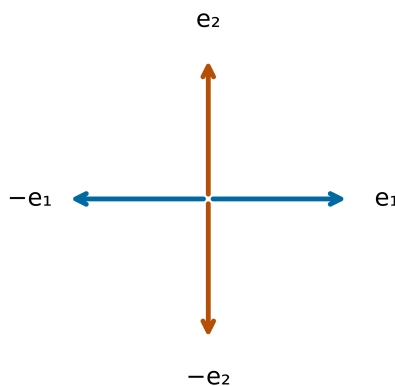
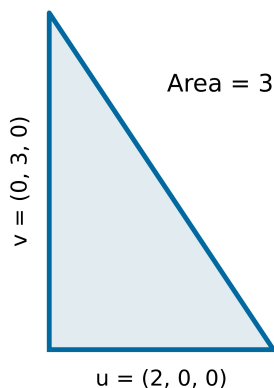
rescue a strict-negative requirement, because its dot products vanish.

Worked reasoning

With $p=(0,0,0)$, $q=(2,0,0)$, $r=(0,3,0)$, the edges are $u=(2,0,0)$, $v=(0,3,0)$. Their cross product is $(0,0,6)$, giving triangle area 3. For $w=(3,4)$ and $u=(1,0)$, the vector projection is $(3,0)$ and the orthogonal remainder is $(0,4)$. The plane $2x-y+2z=6$ has normal length 3, so its distance from the origin is $6/3=2$.

$$\text{Triangle area} = |u \times v| / 2 = 3$$

Two negative products; four zeros



Original course illustration; its values belong to the worked example, not the later practice.

Conditions and counterexamples

A cross-product magnitude is parallelogram area, so halve it for a triangle. Dot-product zero is an algebraic orthogonality statement even for a zero vector. Plane distance must divide by normal length.

Guided application

Project $v = (2, 3, 6)$ onto $u = (1, 0, 2)$. Find the area of the triangle with vertices O , u and v . Check the projection residual is orthogonal to u .

■ Worked solution

$v \cdot u = 14$ and $u \cdot u = 5$, so the vector projection is $(14/5)u = (14/5, 0, 28/5)$. The residual is $(-4/5, 3, 2/5)$, whose dot product with u is zero. The cross product is $u \times v = (-6, -2, 3)$ with norm seven. Triangle area is half the parallelogram area, or $7/2$. Both edge vectors start at O ; crossing unrelated position vectors would not generally compute the requested triangle.

Independent transfer

Can four nonzero vectors in $\mathbb{R}^2$ have all six pairwise dot products strictly negative? Give a proof. How does allowing zero products change the possibilities?

■ Check after attempting

Order their directions around the circle. Every consecutive gap must exceed $\pi/2$: if a gap were at most $\pi/2$, that pair would have nonnegative dot product. Four such gaps would sum to more than 2π , impossible. Therefore not all six products can be strictly negative. With nonpositive products, $e_1, e_2, -e_1, -e_2$ works: consecutive pairs are perpendicular and opposite pairs have negative products. A zero vector cannot satisfy strict negativity.