

C.8 · Trigonometric phase and parametric curves

GRE · GRE Subject Test · GRE Mathematics

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Radian trigonometry, derivatives and curve parametrisation.

- Distinguish amplitude, angular frequency, phase angle and horizontal shift
- Eliminate a parameter while retaining its domain and tracing direction
- Use parametric derivatives without assuming a vertical tangent is stationary

phase angle : The angle offset inside a periodic function argument, defined modulo a full period.

parametric curve : A curve whose coordinates are specified as functions of a shared parameter.

Choose and justify a method

For $y = A \cos(\omega x - \varphi)$ with $A > 0$ and $\omega > 0$, amplitude is A , period is $2\pi/\omega$, and horizontal shift is φ/ω . The phase angle φ is measured inside the cosine argument; it is not itself the horizontal shift unless $\omega = 1$. Start with peak-to-peak spacing for the period and midline-to-peak distance for amplitude. To find phase, substitute a known point and check the direction of motion there. Equivalent phases differ by 2π .

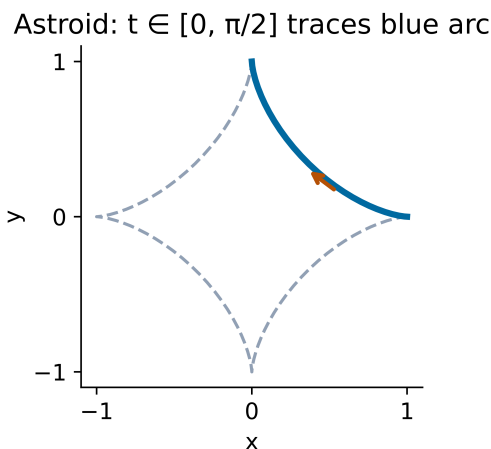
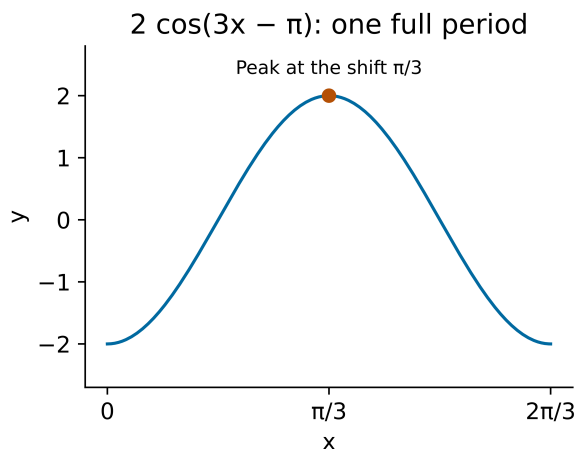
For $y = -2 \cos(3x)$, rewrite the model with positive amplitude as $2 \cos(3x - \pi)$. Its amplitude is 2, period $2\pi/3$, and phase π ; the equivalent right shift is $\pi/3$. Peaks occur where $3x - \pi$ is a multiple of 2π . A graph point at $x=0$ and $y=0$ alone cannot determine phase uniquely: slopes or another point distinguish the possible angles. Units and angle conventions matter; ordinary calculus trigonometric derivatives use radians.

A parametric curve specifies $x=x(t)$, $y=y(t)$. Eliminating t describes a point set but can lose restrictions and direction. For $x=\cos^3 t$, $y=\sin^3 t$, real cube roots give $|x|^{2/3} + |y|^{2/3} = 1$, an astroid with cusps on the axes. As t runs from 0 to 2π it starts at $(1,0)$, passes $(0,1)$ at $\pi/2$ and travels counterclockwise. Restricting t to $[0, \pi/2]$ gives only the first-quadrant arc, not the entire implicit locus.

When $dx/dt \neq 0$, $dy/dx = (dy/dt)/(dx/dt)$. A horizontal tangent generally requires $dy/dt = 0$ and $dx/dt \neq 0$; a vertical tangent generally reverses those conditions. If both vanish, inspect a limit or the local expansion rather than taking $0/0$ as a slope. For $x=t^2, y=t^3$ at $t=0$, the quotient for $t \neq 0$ is $3t/2$ and tends to zero: the cusp has a horizontal tangent. A zero parameter velocity is not by itself a local maximum or minimum of y as a function of x .

Worked reasoning

For $y = -2 \cos(3x)$, minima include $x=0$ and maxima include $x=\pi/3$. The phase representation $2 \cos(3x-\pi)$ gives the same values. For $x=2 \cos t, y=\sin t$, elimination gives $x^2/4+y^2=1$; at $t=\pi/4$, $dx/dt=-\sqrt{2}$ and $dy/dt=\sqrt{2}/2$, hence slope $-1/2$. The full parameter interval $[0, 2\pi]$ traces the ellipse once counterclockwise.



Original course illustration; its values belong to the worked example, not the later practice.

Conditions and counterexamples

Phase angle and horizontal shift differ by the frequency factor. An implicit equation can add untraced parts of a curve. If both parameter derivatives vanish, use local reasoning instead of labelling every such point an extremum.

Guided application

For $y = -3 \cos(2x)$, state amplitude, period, a positive-amplitude phase representation and the corresponding horizontal shift. Parameterise the ellipse $x^2/9 + y^2/4 = 1$ counterclockwise.

■ Worked solution

Amplitude is three and period $2\pi/2 = \pi$. Write $y = 3 \cos(2x - \pi)$, with phase π and rightward shift $\pi/2$. Phase and shift are not the same. One parametrisation is $x = 3 \cos t, y = 2 \sin t, 0 \leq t \leq 2\pi$, which begins at $(3, 0)$ and initially moves upward. Eliminating t checks the ellipse equation.

Independent transfer

For $x = t^2, y = t^3$ on $-1 \leq t \leq 1$, determine the traced locus and tangent at $t=0$. Does vanishing parameter velocity prove there is no tangent? Does the implicit equation alone retain the whole parameter restriction?

■ Check after attempting

Elimination gives $y^2 = x^3$, with $0 \leq x \leq 1$ and both signs of y . For t nonzero, $dy/dx = 3t/2 \rightarrow 0$. The secant slope from the origin is $y/x = t \rightarrow 0$ too, so the cusp has horizontal tangent. Both parameter derivatives vanish at zero, making the raw quotient $0/0$ inconclusive, not proving absence of a tangent. The unrestricted implicit equation also has points with $x > 1$; the interval restriction must be carried separately.