

T.7 • Set images, equivalence relations and logical negation

GRE • GRE Subject Test • GRE Mathematics

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Set operations, quantifiers, functions and elementary proof.

- Prove image identities and distinguish inclusion from equality
- Check reflexivity, symmetry and transitivity with explicit cases
- Negate implications and quantified statements without changing their scope

preimage : The set of inputs whose outputs lie in a specified target set.

equivalence relation : A relation that is reflexive, symmetric and transitive.

Choose and justify a method

For $f:X \rightarrow Y$ and $A \subseteq X$, the image $f(A)$ consists of all outputs $f(a)$ with a in A . A point is in $f(A \cup B)$ exactly when it comes from A or B , so $f(A \cup B) = f(A) \cup f(B)$. If $A \subseteq B$, then $f(A) \subseteq f(B)$. For an intersection only $f(A \cap B) \subseteq f(A) \cap f(B)$ is automatic: the same output may come from different inputs. Use $f(x) = x^2$, $A = \{-1\}$, $B = \{1\}$; the left image is empty while the right intersection is $\{1\}$.

Preimages behave differently. For $C \subseteq Y$, $f^{-1}(C)$ here denotes the set of all inputs sent into C , even if f has no inverse function. Membership in two preimages means the very same input maps into both target sets. Consequently preimages preserve unions, intersections and complements relative to the stated domain/codomain. Do not transfer a theorem about preimages to images. If f is injective, image intersections do become equal, because equal outputs then force a shared input.

A relation R on X is reflexive when xRx for every x , symmetric when xRy implies yRx , and transitive when xRy and yRz imply xRz . All three make an equivalence relation; its classes partition X . On integers, xRy when x and y have the same remainder modulo four gives four classes. The relation $|x - y| \leq 1$ is reflexive and symmetric but not transitive: $0R1$ and $1R2$ while 0 is not related to 2 . Checking only a diagram or two properties is insufficient.

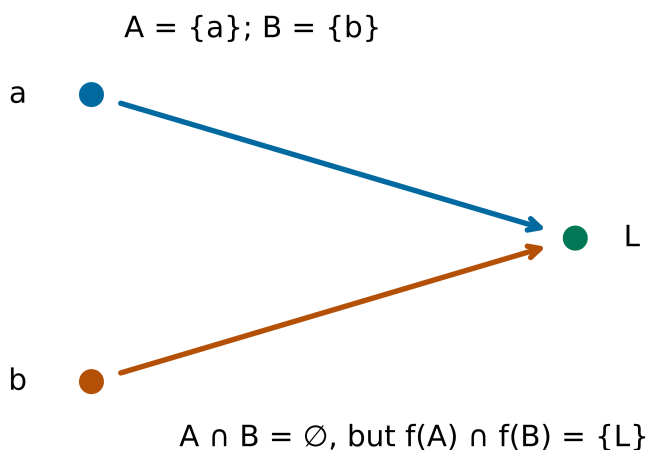
An implication $P \Rightarrow Q$ is false exactly when P is true and Q false. Thus the negation of $P \Rightarrow (Q \wedge R)$ is $P \wedge (\neg Q \vee \neg R)$, not $\neg P \Rightarrow (\neg Q \wedge \neg R)$. Negating “every x has property A ” gives “there exists x without A ”; negating “there exists x ” gives “every x does not”. A counterexample can disprove a universal claim, while examples cannot prove it. The quantifiers retain their order when individually negated: $\neg(\forall x \exists y S(x,y))$ is

$$\exists x \forall y \neg S(x,y).$$

Worked reasoning

Let f map both a and b to label L . With $A=\{a\}$, $B=\{b\}$, $f(A \cap B)=\emptyset$ but $f(A) \cap f(B)=\{L\}$. For xRy defined by $x=y$ or $x=-y$ on $\mathbb{R}$, reflexivity and symmetry follow directly, and two sign changes still give $z=\pm x$, proving transitivity. The classes are $\{x, -x\}$, with $\{0\}$ a singleton. A false statement “every stored file has a hash” means at least one stored file has no hash; it does not mean all files lack hashes.

Two distinct inputs can share an output



Original course illustration; its values belong to the worked example, not the later practice.

Conditions and counterexamples

$f^{-1}(C)$ can mean a preimage set without an inverse function. A reflexive, symmetric relation may still fail transitivity. Negate the whole statement before simplifying its parts.

Guided application

Prove $f(A \cap B) \subseteq f(A) \cap f(B)$. Give a strict-inclusion example. Under what additional assumption on f does equality follow for all A and B ?

■ Worked solution

If $y = f(x)$ for $x \in A \cap B$, then x belongs to each set, so y belongs to each image. For strictness, map distinct a, b to the same c and take $A = \{a\}$, $B = \{b\}$. The left image is empty and the right intersection is $\{c\}$. If f is injective and $y = f(a) = f(b)$ with $a \in A, b \in B$, injectivity gives $a=b$ in the intersection, proving reverse inclusion.

Conversely the singleton example shows that equality for all pairs forces injectivity.

Independent transfer

On the integers define aRb when $a - b$ is divisible by 4. Prove it is an equivalence relation and describe every class. Negate: “For every integer n there exists an integer m greater than n with property P .”

■ Check after attempting

Reflexivity uses $a - a = 0$; symmetry uses the negative of a multiple of four; transitivity uses the sum of two such multiples. Classes are the four residues modulo four; the class of a is $a + 4\mathbb{Z}$. The negation is: there exists an integer n such that every integer m greater than n fails P . Equivalently $\exists n \forall m (m > n \Rightarrow \neg P(m))$. It does not assert that P fails for every integer, nor that m is at most n for every m .