

C.7 · Coordinate changes and vector integral theorems

GRE · GRE Subject Test · GRE Mathematics

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Multiple integrals, partial derivatives and oriented boundaries.

- Transform double and triple integrals with their Jacobians
- Apply Green, Stokes and divergence theorems with correct orientation
- Check domain singularities before asserting path independence

divergence : The sum of a vector field's coordinate-wise partial derivatives measuring local outward flow.

conservative field : A vector field equal to a scalar potential gradient with path-independent line integrals.

Choose and justify a method

Changing variables scales area or volume by the absolute Jacobian determinant. Polar coordinates use $r \, dr \, d\theta$; cylindrical coordinates use $r \, dr \, d\theta \, dz$; spherical coordinates with ϕ measured from the positive z -axis use $\rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta$. State angle conventions and transform both the integrand and the region. A missing Jacobian changes a uniform density integral into the wrong physical quantity.

Green's theorem equates positively oriented planar boundary circulation integral $\int P \, dx + Q \, dy$ with the double integral of $Q_x - P_y$ on the region. The usual hypotheses require first derivatives continuous on an open set containing the region. A hole needs its own negatively oriented inner boundary, or another valid treatment of the missing domain. The theorem cannot integrate across a field singularity.

Stokes' theorem equates circulation on a surface boundary with the surface integral of $\text{curl } \mathbf{F} \cdot \mathbf{n}$ the oriented normal. The right-hand rule links boundary direction to the normal. The divergence theorem equates outward flux across a closed surface with the volume integral of $\text{div } \mathbf{F}$. Circulation, flux, curl and divergence are distinct; a closed surface is required for the usual divergence theorem.

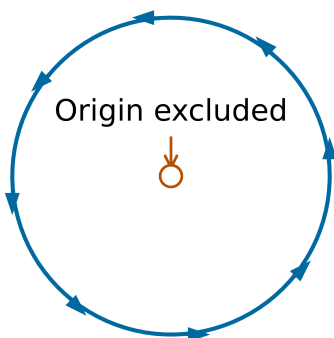
A gradient field has path-independent line integrals, determined by endpoint potential differences. A continuously differentiable curl-free field on a simply connected open domain is conservative. Curl-free alone on a domain with a hole is insufficient. For $\mathbf{F} = (-y/(x^2+y^2), x/(x^2+y^2))$, the origin is excluded; unit-circle circulation is 2π , although the curl is zero wherever the field is defined.

Worked reasoning

For $F=(x,y,z)$, divergence is 3. The outward flux through the sphere of radius 2 is therefore 3 times its volume, or $3 \cdot (4\pi \cdot 2^3/3)=32\pi$. This avoids a surface parameterisation. For the planar field $(-y,x)$, Green's theorem gives counterclockwise unit-circle circulation as integral of $1-(-1)=2$ over the disk, hence 2π . Reversing the orientation changes the circulation sign.

A hole can prevent a global potential

$$F = (-y/r^2, x/r^2)$$



Unit-circle circulation = 2π , despite zero curl

Original course illustration; its values belong to the worked example, not the later practice.

Conditions and counterexamples

Zero curl is not enough when the domain has a hole. Outward flux and counterclockwise circulation use different theorems and orientation rules.

Guided application

Evaluate $\iint_D (x^2 + y^2) dA$ on $1 \leq x^2 + y^2 \leq 4$. Find the outward flux of $F = (x, y, z)$ across the unit sphere.

■ Worked solution

The annulus has polar bounds $1 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$. Including the Jacobian,

$$I = \int_0^{2\pi} \int_1^2 r^3 dr d\theta = 15\pi/2.$$

F is continuously differentiable on the whole ball and has divergence three. The divergence theorem gives outward flux $3(4\pi/3) = 4\pi$. Inward orientation would reverse its sign.

Independent transfer

Let $G = (-y/(x^2+y^2), x/(x^2+y^2))$ on the punctured plane. Its scalar curl is zero there. Evaluate its circulation on the counterclockwise unit circle. Explain why Green's theorem does not make it zero.

■ Check after attempting

Parameterise $r(t) = (\cos t, \sin t)$ for $0 \leq t \leq 2\pi$. Then $G(r(t)) = (-\sin t, \cos t) = r'(t)$, so $G \cdot r' = 1$ and the integral is 2π . The field is undefined at the origin inside the disk. Green's theorem requires the needed smoothness on a neighbourhood of the region, not just on its boundary. Zero curl on a punctured domain does not establish a global potential; a closed nonexact field can circulate around its hole.