

C.6 · Multivariable extrema and constrained optimisation

GRE · GRE Subject Test · GRE Mathematics

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Partial derivatives, quadratic forms, compactness and constraints.

- Classify two-variable critical points with the Hessian
- Use Lagrange multipliers with a regular constraint
- Compare extrema and attainable values under regular constraints

Hessian : The matrix of second partial derivatives used to study local curvature.

Lagrange multiplier : A scalar relating objective and regular constraint gradients at a constrained extremum.

Choose and justify a method

For a differentiable function on an open domain, an interior local extremum has zero gradient. This condition is necessary, not sufficient. For two variables at a critical point, let $D = f_{xx}f_{yy} - (f_{xy})^2$. If $D > 0$, $f_{xx} > 0$ gives a strict local minimum and $f_{xx} < 0$ gives a strict local maximum; $D < 0$ gives a saddle. These tests require suitable second-derivative regularity near the point.

When $D = 0$ the second-derivative test is inconclusive, not evidence of a saddle. The functions $x^4 + y^4$ and $x^4 - y^4$ have the same zero Hessian at the origin, but one has a strict minimum and the other changes sign. Evaluate along contrasting directions or use a direct inequality. For a quadratic form, positive definiteness gives a more general Hessian interpretation.

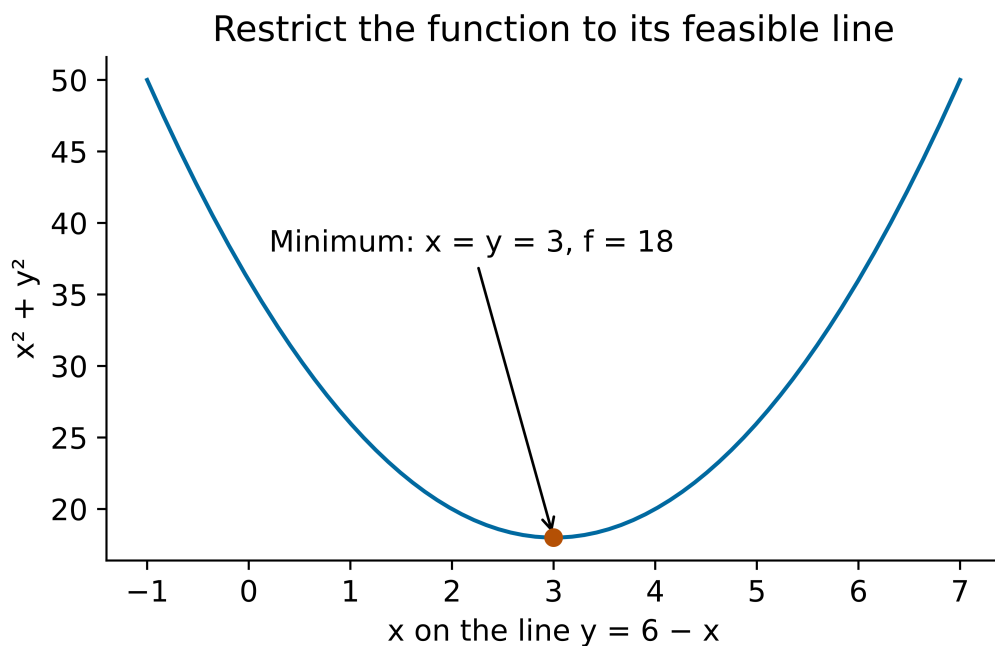
On a smooth equality constraint $g(x, y) = c$ with nonzero gradient g , solve gradient $f = \lambda \text{gradient } g$ together with $g = c$. These equations generate candidates; they do not classify or guarantee a global extremum. If the constraint gradient vanishes, the regularity condition fails and the multiplier equations can miss a constrained extremum. Treat such points separately. For $x^2 + y^2 = a$ and $xy = b > 0$, $(x - y)^2 = a - 2b$ shows necessity $a \geq 2b$. It is also sufficient: take $s = \sqrt{a + 2b}$, $d = \sqrt{a - 2b}$, $x = (s + d)/2$ and $y = (s - d)/2$. These real values have $xy = b$ and $x^2 + y^2 = a$. A minimum condition alone needs this attainment check to establish solvability.

A continuous function on a compact feasible set attains global extrema. To find them, compare all interior candidates and all boundary pieces, including corners and endpoints. For a rectangle, optimise the restrictions on each edge. For a disk, the circular boundary may use a parameter or multiplier. Solving only the unconstrained

gradient ignores possible boundary winners.

Worked reasoning

Minimise $x^2 + y^2$ subject to $x + y = 6$. The constraint gradient is $(1, 1)$, so $2x = \lambda$ and $2y = \lambda$. Thus $x = y$ and the constraint gives $x = y = 3$, with value 18. Completing the square on the line gives $x^2 + (6 - x)^2 = 2(x - 3)^2 + 18$, proving a global minimum. There is no maximum on this unbounded line, despite the existence of a multiplier candidate.



Original course illustration; its values belong to the worked example, not the later practice.

Conditions and counterexamples

A Lagrange multiplier solution is a candidate, not automatically a maximum or minimum. Check regularity and the complete feasible set.

Guided application

Classify the origin for $f(x, y) = x^2 - y^2$ and for $g(x, y) = x^4 + y^4$. State what a zero Hessian determinant permits you to conclude.

■ Worked solution

For f the Hessian is $\text{diag}(2, -2)$ with determinant -4 ; the origin is a saddle. Along the axes the values have opposite signs. For g the Hessian at zero is zero, so the second-derivative test is inconclusive. Nevertheless $x^4 + y^4 \geq 0$, with equality only at the origin; it is a strict global minimum. An inconclusive test is not a classification of the point.

Independent transfer

Find all extrema of $f(x, y) = xy$ subject to $x^2 + y^2 = 2$. Compare with minimising $x^2 + y^2$ on the line $x + y = 4$, including whether maxima exist.

■ Check after attempting

The circle is compact and its constraint gradient $(2x, 2y)$ never vanishes there. Multipliers give $y = 2\lambda x$, $x = 2\lambda y$. Neither coordinate can be zero at a stationary point; hence $y = \pm x$. The four candidates $(1, 1), (-1, -1), (1, -1), (-1, 1)$ give global maximum 1 and minimum -1. The bound $2|xy| \leq x^2 + y^2 = 2$ checks completeness. On the line, $x^2 + (4 - x)^2 = 2(x - 2)^2 + 8$, so the minimum is 8 at $(2, 2)$. There is no maximum because the expression is unbounded. Multiplier candidates alone do not establish compactness.