

T.4 · Sequences, series and uniform convergence

GRE · GRE Subject Test · GRE Mathematics

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Sequence limits, supremum, continuity and numerical series.

- Use Cauchy, monotone convergence and subsequence criteria
- Distinguish pointwise from uniform convergence of functions
- Check hypotheses before interchanging limits and integrals

Cauchy sequence : A sequence whose sufficiently late terms are arbitrarily close to one another.

uniform convergence : Convergence with one error threshold index valid at every point of the domain.

Choose and justify a method

A convergent real sequence is Cauchy, and every real Cauchy sequence converges because $\mathbb{R}$ is complete. In $\mathbb{Q}$ a Cauchy sequence can approach an irrational number and fail to converge within $\mathbb{Q}$. A bounded monotone real sequence converges. A bounded sequence need not converge, but Bolzano–Weierstrass guarantees a convergent subsequence; $(-1)^n$ has two different subsequential limits.

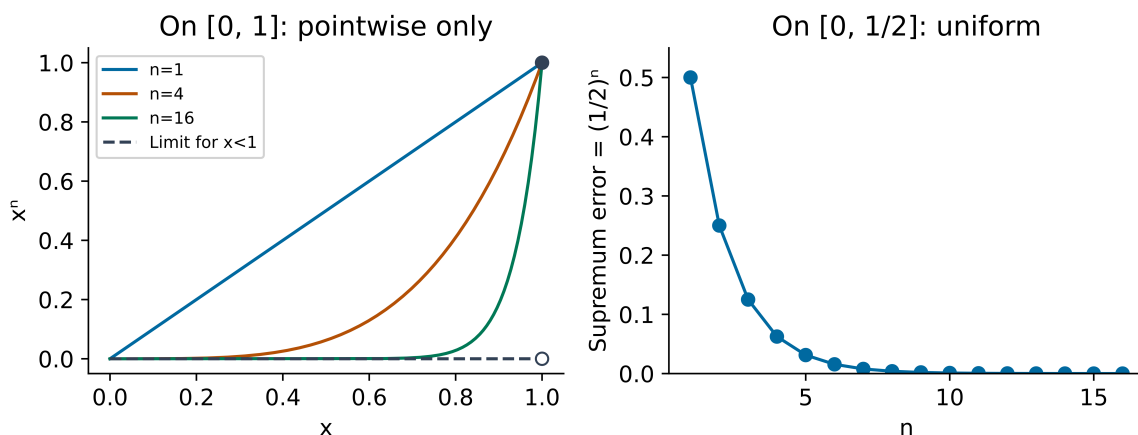
For a numerical series, absolute convergence implies convergence; conditional convergence does not allow arbitrary rearrangement without affecting the sum. The alternating harmonic series converges but its absolute-value series diverges. In a ratio test, a limit below 1 proves absolute convergence and one above 1 proves divergence; a limit equal to 1 is inconclusive, as both $\sum 1/n$ and $\sum 1/n^2$ demonstrate.

Pointwise convergence chooses an index N separately for each x and tolerance. Uniform convergence chooses one N that works for every x in the domain. For real-valued functions, check the supremum of $|f_n - f|$ over the whole domain. A continuous pointwise limit does not by itself prove uniform convergence. Domain endpoints and shrinking peaks often distinguish the two notions.

A uniform limit of continuous functions is continuous. On a closed bounded interval, uniform convergence of Riemann-integrable functions permits exchanging limit and integral. Exchanging derivatives needs extra hypotheses; uniform convergence of the functions alone is insufficient. The Weierstrass M-test establishes uniform absolute convergence of a series of functions if each term is bounded by M_n on the whole domain and $\sum M_n$ converges.

Worked reasoning

On $[0,1]$, $f_n(x)=x^n$ tends to 0 for $x<1$ and to 1 at $x=1$. The limit is discontinuous, so convergence cannot be uniform; directly, the supremum error is 1, approached below 1. On $[0,a]$ with $0\leq a<1$, the limit is zero and the supremum is a^n , which tends to zero, so convergence is uniform. Changing the domain changes the conclusion.



Original course illustration; its values belong to the worked example, not the later practice.

Conditions and counterexamples

Checking several fixed x values proves neither a supremum bound nor uniform convergence. The point producing the largest error may move with n .

Guided application

Determine pointwise and uniform convergence of $f_n(x) = x^n$ on $[0, 1]$ and on $[0, a]$ for fixed $0 < a < 1$. Compare $\sum 1/n^2$ with the alternating harmonic series for absolute convergence.

■ Worked solution

On $[0, 1]$, the limit is zero for $x < 1$ and one at 1. This limit is discontinuous, so continuous f_n cannot converge uniformly. More directly the supremum error on $[0, 1)$ is one. On $[0, a]$ the supremum is $a^n \rightarrow 0$, so convergence is uniform to zero. The p-series with $p=2$ converges absolutely. The alternating harmonic series converges by decreasing terms tending to zero, but the absolute series diverges; its convergence is conditional.

Independent transfer

Let $g_n(x) = \sin(nx)/n$ on $\mathbb{R}$. Is convergence uniform? May its derivatives converge to the derivative of its limit? Use an explicit point to decide.

■ Check after attempting

$\sup_{x \in \mathbb{R}} |g_n(x)| = 1/n \rightarrow 0$, so the functions converge uniformly to zero. But $g'_n(x) = \cos(nx)$, and at $x = 0$ these derivatives are all 1. The derivative of the zero limit is 0. Thus uniform convergence of the functions alone does not justify differentiating a limit; additional derivative-convergence hypotheses are needed.