

A.4 · Rings, ideals and modules

GRE · GRE Subject Test · GRE Mathematics

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Rings, fields, additive subgroups and homomorphisms.

- Distinguish units, zero divisors and integral domains
- Identify ideals and interpret quotient rings
- Compare modules over rings with vector spaces over fields
- Derive Boolean-ring properties without assuming commutativity

ideal : An additive subgroup of a ring absorbing multiplication by all ring elements.

torsion : An element is annihilated by a nonzero scalar in the stated module.

Choose and justify a method

For the domain, ideal and quotient examples below, use a commutative ring with identity 1 distinct from 0. A source problem may specify a general ring instead; do not assume its multiplication commutes unless stated or proved. A unit has a multiplicative inverse. A nonzero zero divisor multiplies some nonzero element to zero. An integral domain has no such zero divisors; cancellation of a nonzero factor then works. A field is a domain in which every nonzero element is a unit. $\mathbb{Z}$ is a domain but not a field; $\mathbb{Z}/6\mathbb{Z}$ is neither.

An ideal I is an additive subgroup that absorbs multiplication by every ring element. This is stronger than being a subring. In $\mathbb{Z}$, $n\mathbb{Z}$ is an ideal; quotient elements are integer residue classes modulo n . In a commutative ring, R/I is a field exactly when I is maximal, and it is a domain exactly when I is prime. The ideal must be proper in both statements.

A module allows scalars from a ring instead of requiring a field. Every abelian group is a $\mathbb{Z}$ -module by repeated addition, but it need not have a vector-space basis. In $\mathbb{Z}/6\mathbb{Z}$ as a $\mathbb{Z}$ -module, 6 times the nonzero residue 1 is zero; this is torsion. For a vector space over a field, a nonzero scalar is invertible and cannot annihilate a nonzero vector.

A submodule is closed under addition and all permitted scalar actions. A linear map of modules preserves both. The kernel and image are submodules, and the quotient by the kernel is isomorphic to the image. Do not apply finite-dimensional rank-nullity to an arbitrary module without establishing an appropriate free-module setting; integer row operations and field row operations permit different divisions. In a general Boolean ring, every a satisfies $a^2=a$. Do not assume commutativity to prove it: idempotence of $a+a$ gives $4a=2a$, hence $2a=0$. Expanding $(a+b)^2=a+b$ gives $ab+ba=0$,

and characteristic two makes $-ba=ba$; therefore $ab=ba$. Idempotence does not imply nilpotence: in F_2 , the nonzero element 1 satisfies $1^n=1$ for every positive n .

Worked reasoning

In $\mathbb{Z}/6\mathbb{Z}$, $2 \cdot 3=0$, so 2 and 3 are zero divisors. The units are 1 and 5 because their gcd with 6 is 1. In $\mathbb{Z}$, the ideal $5\mathbb{Z}$ gives the field $\mathbb{Z}/5\mathbb{Z}$, while $6\mathbb{Z}$ gives a quotient with zero divisors. As $\mathbb{Z}$ -modules, the map from $\mathbb{Z}$ to $\mathbb{Z}/6\mathbb{Z}$ has kernel $6\mathbb{Z}$; the quotient identifies integers differing by a multiple of 6, rather than producing a real vector space.

Multiplication in $\mathbb{Z}/6\mathbb{Z}$: units and zero divisors

Residue	Product mod 6	Classification
1, 5	$1 \cdot 1 = 1$; $5 \cdot 5 = 1$	Units
2, 3, 4	$2 \cdot 3 = 0$; $4 \cdot 3 = 0$	Nonzero zero divisors
0	$0 \cdot a = 0$	Zero

Original course illustration; its values belong to the worked example, not the later practice.

Conditions and counterexamples

A set can be a submodule without being a vector space over the rationals. Scalar division is valid only when the scalar inverse belongs to the structure.

Guided application

In $\mathbb{Z}/12\mathbb{Z}$, list the units and nonzero zero divisors. Is the ideal (4) prime or maximal?

■ Worked solution

Units are the residues coprime to 12: 1, 5, 7, 11. The remaining nonzero residues 2, 3, 4, 6, 8, 9, 10 are zero divisors. For each such a , $a \cdot (12/\gcd(a, 12)) = 0$ modulo 12, with the second factor nonzero. The ideal $(4) = \{0, 4, 8\}$ gives quotient isomorphic to $\mathbb{Z}/4\mathbb{Z}$. This is not a domain, since $2 \cdot 2 = 0$, and is not a field. Hence the ideal is neither prime nor maximal.

Independent transfer

As a $\mathbb{Z}$ -module, can $\mathbb{Z}/6\mathbb{Z}$ be free? Compare the ideals (2) and (x) in $\mathbb{Z}$ and $\mathbb{Q}[x]$, respectively, using their quotients.

■ Check after attempting

A nonzero free $\mathbb{Z}$ -module has no nonzero vector annihilated by a nonzero integer: examine each integer coordinate. But $6[1] = 0$ in $\mathbb{Z}/6\mathbb{Z}$. It is not free; a field-style basis argument is invalid. $\mathbb{Z}/(2) \cong \mathbb{F}_2$ is a field, so (2) is maximal and prime. Evaluation at zero maps $\mathbb{Q}[x]$ onto $\mathbb{Q}$ with kernel (x) ; its quotient is a field too. Both arguments use proper ideals in commutative rings with identity.