

C.3 · Differential equations and initial conditions

GRE · GRE Subject Test · GRE Mathematics

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Differentiation, integration, exponentials and quadratic equations.

- Solve separable first-order equations
- Solve constant-coefficient second-order equations
- Use initial conditions and uniqueness conditions

initial condition : A value of the solution or derivative specified at a starting point.

characteristic root : A root of the polynomial governing exponential solutions.

Choose and justify a method

For $y' = ky$, separation and integration give $y = Ce^{kt}$; recover the zero solution if division by y was used. An initial value determines C .

A linear second-order constant-coefficient equation uses a characteristic polynomial. Distinct roots yield exponentials; a repeated root requires $(C_1 + C_2 t)e^{rt}$.

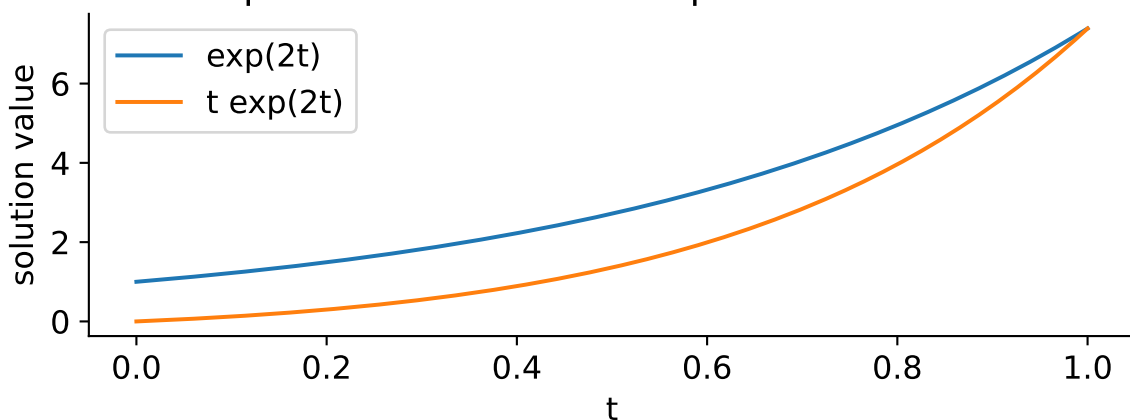
Complex roots $\alpha \pm i\beta$ give $e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$. The real motion includes both amplitude and phase information.

Existence and uniqueness depend on hypotheses near the initial point. A singular coefficient or a failure of local Lipschitz behaviour can defeat the familiar uniqueness conclusion.

Worked reasoning

Solve $y'' + 4y = 0$ with $y(0) = 3$ and $y'(0) = 4$. The characteristic roots are $\pm 2i$, so $y = A \cos 2t + B \sin 2t$. The first condition gives $A = 3$; differentiating gives $y'(0) = 2B = 4$, so $B = 2$.

Repeated root 2: two independent solutions



Original course illustration; its values belong to the worked example, not the later practice.

Conditions and counterexamples

A repeated characteristic root needs the factor t ; two copies of the same exponential are not independent solutions.

Guided application

Solve $y'' - 4y' + 4y = 0$ with $y(0) = 1$, $y'(0) = 0$. Check both initial conditions and the equation.

■ Worked solution

The characteristic polynomial is $(r - 2)^2$. Independent solutions are e^{2t} and te^{2t} , hence $y = (C_1 + C_2t)e^{2t}$. The initial value gives $C_1 = 1$; the derivative at zero gives $2C_1 + C_2 = 0$, so $C_2 = -2$.

$$y(t) = (1 - 2t)e^{2t}, \quad y'(t) = -4te^{2t}, \quad y''(t) = (-4 - 8t)e^{2t}.$$

Substitution gives $y'' - 4y' + 4y = 0$, and the two initial values hold. Two copies of e^{2t} would not span the solution space.

Independent transfer

Show that the initial-value problem $y' = 2\sqrt{|y|}$, $y(0) = 0$ has more than one solution for $t \geq 0$. Identify the missing standard uniqueness hypothesis.

■ Check after attempting

One solution is identically zero. For any $a \geq 0$, set $y_a(t) = 0$ for $0 \leq t \leq a$ and $y_a(t) = (t - a)^2$ for $t > a$. The derivative is zero on the first interval and $2(t - a)$ on the second; both sides have derivative zero at a . Therefore y_a is differentiable and satisfies the equation and initial value. The right side is continuous but is not locally Lipschitz in y at zero: $|2\sqrt{h} - 0|/|h| = 2/\sqrt{h}$ is unbounded. Existence does not imply uniqueness, and division by $\sqrt{y}$ would lose the waiting-time solutions.