

T.3 · Complex analysis and residues

GRE · GRE Subject Test · GRE Mathematics

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Complex arithmetic, partial derivatives, series and oriented contours.

- Test complex differentiability
- Use contour integrals and residues
- Evaluate contour integrals with simple and higher-order pole residues

analytic : Complex differentiable throughout a neighbourhood.

residue : Coefficient of $(z-a)^{-1}$ in a Laurent series.

Choose and justify a method

Write $f(z)=u(x,y)+iv(x,y)$. Complex differentiability imposes $u_x=v_y$ and $u_y=-v_x$; with continuous first partials locally, the Cauchy–Riemann equations establish analyticity there. They differ from real differentiability of a two-coordinate map. The conjugate function $x-iy$ fails these equations on every open neighbourhood, although real partial derivatives exist. State the region being checked, not only a convenient point.

An analytic function is complex differentiable throughout a neighbourhood and has a local convergent power series. A removable singularity can be filled analytically when the function is bounded near the missing point. A pole has a finite principal part in its Laurent series; an essential singularity has infinitely many negative-power terms. The residue is the coefficient of $(z-a)^{-1}$, not necessarily the leading or largest negative-power term.

For an isolated pole of order m , write $f(z)=g(z)/(z-a)^m$ with g analytic at a . The residue is $g^{(m-1)}(a)/(m-1)!$. A simple pole uses $g(a)$; a double pole uses $g'(a)$. This follows by expanding g into its Taylor series. Thus $e^z/(z-a)^2$ has residue e^a , while a constant numerator over a pure double pole has zero residue. The pole order alone does not determine the contour integral.

For a positively oriented contour enclosing isolated singularities, the residue theorem gives $\oint f(z)dz=2\pi i$ times the sum of enclosed residues, provided the function is analytic on the contour and elsewhere in the required interior. Reversing orientation changes the sign. Cauchy’s derivative formula is $\oint g(z)/(z-a)^{(m+1)}dz=2\pi i g^{(m)}(a)/m!$ under its analytic-domain hypotheses. A singularity on the contour prevents direct application; a singularity outside contributes nothing to this contour.

Worked reasoning

For $f(z)=\text{conjugate}(z)$, $u=x$ and $v=-y$. Then $u_x=1$ but $v_y=-1$, so f is not complex differentiable. For $1/(z-2)$, a positively oriented circle $|z-2|=1$ encloses a simple pole of residue 1, hence the integral is $2\pi i$.

The residue is the coefficient of $(z-1)^{-1}$

Function	Pole order	Residue at 1
$3/(z-1)$	1	3
$e^z/(z-1)^2$	2	e
$e^{2z}/(z-1)^3$	3	$2e^2$

Original course illustration; its values belong to the worked example, not the later practice.

Conditions and counterexamples

Continuity or real differentiability alone does not imply complex analyticity.

Guided application

Use the Cauchy–Riemann equations to compare z^2 and $\bar{z}$. Find the residue of e^z/z^2 at zero and its integral around the counterclockwise unit circle.

■ Worked solution

For z^2 , $u = x^2 - y^2$ and $v = 2xy$: $u_x = 2x = v_y$ and $u_y = -2y = -v_x$. The partials are continuous everywhere, so the function is analytic everywhere. For $\bar{z}$, $u = x, v = -y$ and $u_x = 1 \neq -1 = v_y$; it is nowhere complex differentiable. Since $e^z = 1 + z + z^2/2 + \dots$, the coefficient of z^{-1} in e^z/z^2 is 1. Therefore the residue is 1 and the integral is $2\pi i$. The double pole does not force a zero residue.

Independent transfer

Integrate $1/[z(z-2)]$ around $|z|=1$, first counterclockwise and then clockwise. Can the same residue-theorem argument be used on $|z|=2$?

■ Check after attempting

Only zero is inside the unit circle. Its residue is $\lim_{z \rightarrow 0} 1/(z-2) = -1/2$. Thus the counterclockwise integral is $-\pi i$ and the clockwise integral is πi . The pole at 2 is outside and contributes nothing. The circle $|z|=2$ passes through a pole. The ordinary contour integral is not defined by this formula; the theorem requires analyticity on the contour. An indentation or principal-value prescription would be a different, explicitly specified problem.