

T.1 · Real analysis and topology

GRE · GRE Subject Test · GRE Mathematics

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Quantified statements, inequalities, sequences and continuous functions.

- Apply sequence and function limit definitions
- Distinguish compactness, connectedness and completeness
- Use metric-space and elementary topological reasoning

compact : Every open cover has a finite subcover.

supremum : The least upper bound of a set.

Choose and justify a method

An epsilon–delta statement controls all sufficiently close inputs. The quantifier order matters.

In real Euclidean space, closed and bounded sets are compact. Do not apply this equivalence to every metric space.

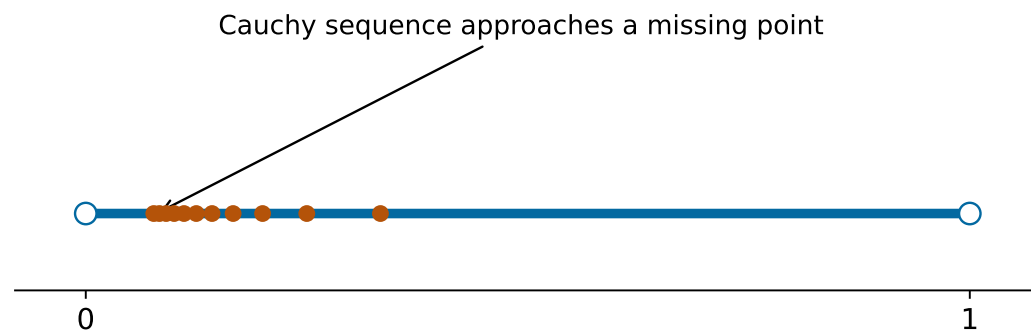
Continuous images of compact sets are compact, so a real continuous function on a compact domain attains extrema.

Connectedness rules out a separation into disjoint nonempty open parts. Continuity preserves connectedness; completeness is a separate property.

Worked reasoning

$f(x)=x$ on $(0,1)$ is continuous and bounded but never equals its supremum 1. On $[0,1]$, the same function attains its maximum at 1. The missing endpoint explains why the compact-domain theorem does not apply to the first case.

$X = (0, 1)$: bounded, but incomplete



Original course illustration; its values belong to the worked example, not the later practice.

Conditions and counterexamples

A theorem’s conclusion cannot be used before its hypotheses have been checked.

Guided application

Prove continuity of $f(x) = x^2$ at 2 using an explicit epsilon–delta choice. Explain separately why it attains a minimum on $[1, 3]$.

■ Worked solution

Given $\epsilon > 0$, choose $\delta = \min(1, \epsilon/5)$. If $|x - 2| < \delta$, then $1 < x < 3$, so $|x + 2| < 5$.

$$|x^2 - 4| = |x - 2||x + 2| < 5\delta \leq \epsilon.$$

The choice depends on epsilon, not on the later input x . The interval $[1, 3]$ is nonempty and compact in the ordinary real metric. Continuity gives attained extrema. Here $x^2 \geq 1$, with equality at 1, so the minimum is 1.

Independent transfer

In the metric space $X = (0, 1)$ with ordinary distance, is X bounded? Is it closed in itself? Is it compact or complete? Explain why “closed and bounded implies compact” cannot be used here.

■ **Check after attempting**

X is bounded and closed relative to itself, since its complement in X is empty. The Cauchy sequence $1/(n+2)$ has no limit in X , so X is not complete and therefore not compact as a metric space. Directly, the relative open cover $\{(1/n, 1) : n \geq 2\}$ covers X but has no finite subcover. Heine–Borel requires closed and bounded as a subset of Euclidean space, not merely closed in an arbitrary chosen space. X is not closed in $\mathbb{R}$. No theorem hypothesis may change its ambient space halfway through the argument.